Poincaré inequality:

$f : \{-1,1\}^n \to \{0,1\}$

$I_i(f) := \mathbb{E}|\partial_i f|$ $\qquad$ $\partial_i f(x) := f(x_{i\to 1}) - f(x_{i\to -1})$.

Poincaré: $\quad \mathrm{Var}[f] \leq \sum_i I_i(f)$.

Multilinear extesion:

$$f(x) = \sum_{A\subset[n]} \hat{f}(A) \prod_{i\in A} x_i$$

defined for all $x \in [-1,1]^n$.

It's a <u>harmonic</u> fn. $\left(\forall x,\ \sum_i \frac{\partial^2 f}{\partial_i^2}(x) = 0\right)$.

Define $\quad X_0, X_1, \ldots X_n \in \{-1,0,1\}^n$.

$k_1, \ldots, k_n$ random permutation $[n]$.

$\varepsilon_1, \ldots, \varepsilon_n \sim B(\pm 1)$ iid.

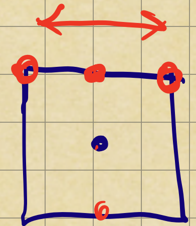
Define $\quad X_i - X_{i-1} = \varepsilon_i \vec{e}_{k_i}$ $\qquad$ ($\vec{e}_1, \ldots \vec{e}_n$ std basis)

$X_n \sim U(\{-1,1\}^n)$

$M_i := f(X_i)$, $\quad \mathrm{Var}[M_n] = \mathrm{Var}[f]$.

$$\frac{1}{n}\sum I_i(f) = \mathbb{E}(M_n - M_{n-1})^2 = \mathbb{E}\left[\left(\partial_{k_n} f(X_n)\right)^2\right]$$

$$= \mathbb{E}\left[I_{k_n}(f)\right]$$



Fact: $(M_i)_i$ is a martingale.

$$\mathbb{E}[M_i \mid X_0, \dots X_{i-1}] = M_{i-1}$$

$$= \mathbb{E}_{k_i}\left[\mathbb{E}[M_i \mid X_0, \dots X_{i-1}, k_i]\right] =$$

$$\mathbb{E}_{k_i}\left[\frac{1}{2} f(X_{i-1} + e_{k_i}) + \frac{1}{2} f(X_{i-1} - e_{k_i})\right]$$

$$= \mathbb{E}_{k_i}\left[f(X_{i-1})\right] = M_{i-1}.$$

$$\mathbb{E}\left[M_i \;\middle|\; \begin{array}{c}\text{all the past}\\ \text{until } M_{i-1}\end{array}\right] = M_{i-1}$$

If (M_i) is a martingale then

$$\mathrm{Var}(M_j - M_i) = \mathbb{E} \sum_{t \in [i,j]} (M_t - M_{t-1})^2 .$$

$$\|$$

$$\mathbb{E}\Big((M_j - M_{j-1}) + (M_{j-1} - M_{j-2}) + \cdots \quad (M_{i+1} - M_i)\Big)^2$$

$$\Rightarrow \quad \mathrm{Var}(f) = \mathrm{Var}\, M_n = \mathbb{E} \sum_{i \in [n]} (M_i - M_{i-1})^2$$

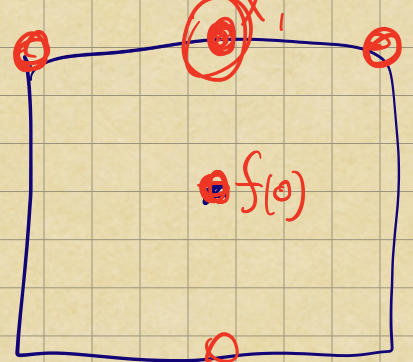
[recall $\quad \mathbb{E}(M_n - M_{n-1})^2 = \frac{1}{n} \sum_i I_i(f)$].

Want:

$$\leq n\,\mathbb{E}(M_n - M_{n-1})^2$$

$$\Uparrow$$

$$\mathbb{E}(M_i - M_{i-1})^2 \leq \mathbb{E}(M_n - M_{n-1})^2 \quad \forall i \in [n]$$



$M_1 = f(x_1)$

$$(M_i - M_{i-1}) = \partial_{k_i} f(X_{i-1}) \cdot \varepsilon_i$$

<u>Fact:</u> $\forall j$, $\partial_j f(X_i)$ is a martingale.

Reason: $\partial_j f$ is also multilinear!

$$\mathbb{E}(M_i - M_{i-1})^2 = \mathbb{E}_{k_i} \mathbb{E}_{X_{i-1}} (\partial_{k_i} f(X_{i-1}))^2$$

??

$$\leq \mathbb{E}_{k_i} \mathbb{E}\left[(\partial_{k_i} f(X_n))^2\right]$$

$$= \frac{1}{n} \sum_k \mathbb{E}(\partial_k f(X_n))^2 = \frac{1}{n} \sum_i I_i(f).$$

$N_{i,t} := \partial_{k_i} f(X_t)$ martingale w.r.t. t.

$\Rightarrow$ $\mathbb{E}(N_{i,t})^2$ increasing.

$N_1, \ldots N_n$ $\qquad \mathbb{E}[N_i \mid N_1, \ldots N_{i-1}] = N_{i-1}$

φ convex

$$\mathbb{E}[\varphi(N_i) \mid N_1, \ldots N_{i-1}] \overset{\text{Jensen}}{\geq} \varphi(N_{i-1}).$$

Sampling a random bit:

*Toss a $\left(\frac{1}{2}, \frac{1}{2}\right)$ coin.

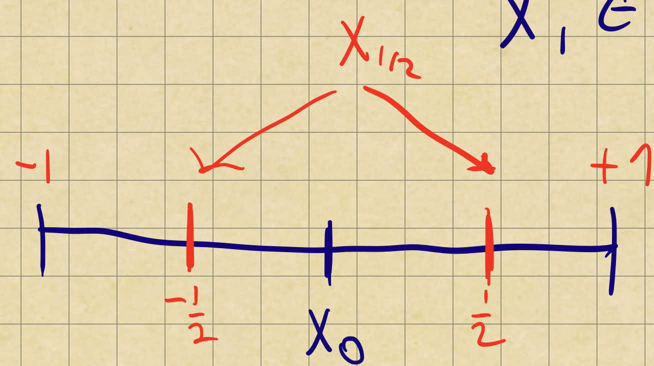
- heads: toss $\left(\frac{3}{4}, \frac{1}{4}\right)$ coin
  - heads $= 1$
  - tails $= -1$
- tails: toss $\left(\frac{1}{4}, \frac{3}{4}\right)$ coin
  - heads $= +1$
  - tails $= -1$.

Every such scheme

$\Longleftrightarrow$

Martingale X_t s.t $X_0 = 0$

$X_1 \in \{-1, 1\}$.



$$X_{1/2} = \begin{cases} -\frac{1}{2} & \text{w.p } \frac{1}{2} \\ +\frac{1}{2} & \text{w.p } \frac{1}{2} \end{cases}$$

If q unique sol to $\frac{1+p}{2} \log \frac{1+p}{2} + \frac{1-p}{2} \log \frac{1-p}{2}$

$= \frac{1}{2} \log 2.$

$$X_{\frac{1}{2}} = \begin{cases} 9 & \text{w.p } \frac{1}{2} \\ -9 & \text{w.p } \frac{1}{2} \end{cases}$$

$$H(X_{1/2}) = \frac{1}{2} H(X_1).$$

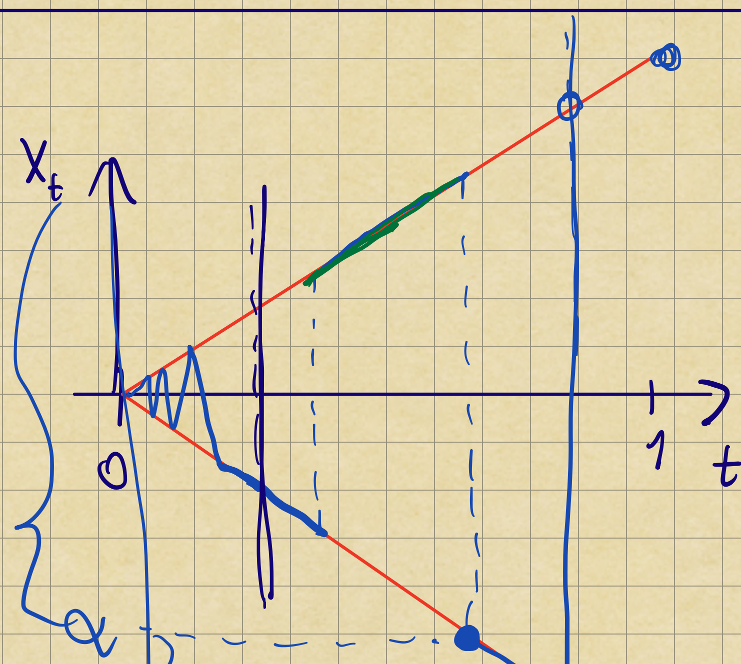
Two "continuous" ways to sample a bit:

① $\exists$ process X_t s.t:

1. X_t is a martingale
2. $|X_t| = t$.

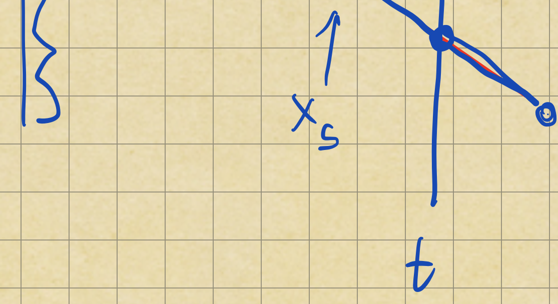
② $X_t = B_{t \wedge \tau}$ $\quad \tau = \min\{s \mid B_s \in \{\pm 1\}\}$

B Brownian motion.



$\int \frac{1}{2t}$ diverges near 0

$\mathbb{E}[X_1 \mid X_t] = X_t$



$\tau(t) := \min\{s ; |B_s| = t\}$

$X_t := B_{\tau(t)}$.

Suppose:

$X_t = +t$ for some t.

Consider distribution of $X_{t+dt} \mid X_t$

[from now on we'll fix a small number "dt"
$t \in \mathbb{Z}\, dt$]

$$X_{t+dt} = \begin{cases} +t+dt & \text{w. p. } 1-p \\ -t+dt & \text{w.p } p \end{cases}$$

$p = ?$

(X_t) jumps between time t and $t+dt$

$\mathbb{E}[X_{t+dt} \mid X_t] = X_t = t$

$\|$

$(1-p)(t+dt) + p\cdot(-(t+dt))$

$\boxed{p = \frac{dt}{2t}}$ $\cancel{(1+O(dt))}$

$$\frac{dt}{2t}\cdot(-2t) + (1-O(dt))(+dt) = O((dt)^2).$$

Suppose $X(t)$ is any martingale

$[-1,1]^n$ s.t $X(0)=0$, $|X(1)|=1$

Let $\vec{X}(t)$ be s.t $X_i(t)$ are iid with

law of $(X(t))_t$.

For f mult. lin extension of a Boolean fn.

Define $M_t = f(\vec{X}(t))$

Fact: M_t is a martingale.

Reason:

$$f(\vec{X}(t)) = \sum_{A\subseteq[n]} \hat{f}(A) \prod_{i\in A} X_i(t)$$

For all A, $\prod_{i\in A} X_i(t)$ is a prod.

of independent martingales.

Want some notion of $\sum_{i\in[n]} (M_i - M_{i-1})^2$.

Quadratic variation:

$$[M]_t := \lim_{\varepsilon\to 0} \sum_{s\in[0,t]\cap\varepsilon\mathbb{Z}} (M_s - M_{s+\varepsilon})^2.$$

Itô's isometry:

$$\mathrm{Var}[M_t - M_s] = \mathbb{E}\left([M]_s - [M]_t\right). \quad \forall s>t$$

$t=1,\ s=0$

$$\mathrm{Var}[f] = \mathbb{E}[M]_1 = \int_0^1 \left(\mathbb{E}[M]_{t+dt} - \mathbb{E}[M]_t\right)$$

$$= \int_0^1 \left[\mathbb{E}(M_{t+dt} - M_t)^2\right] = \int_0^1 \mathbb{E}\left[\left(f(\vec{X}(t+dt)) - f(\vec{X}(t))\right)^2\right]$$

* If no coords jumped $\Rightarrow \|X(t+dt) - X(t)\|$ is $O(dt)$

$$\Rightarrow |f(X(t+dt)) - f(X(t))| = O(dt)$$

So $(\qquad)^2 = o(dt)$.

* The prob. that more than one coord jumps between $t, t+dt$ is $o(dt)$

$$= \mathbb{E} \int_0^1 \sum_{i \in [n]} \overbrace{\mathbb{P}\begin{pmatrix} X_i(t) \text{ jumped} \\ \text{between } t, t+dt \end{pmatrix}}^{\frac{dt}{2t}} \left(\partial_i f(X_t) \cdot 2t\right)^2$$

$$= \mathbb{E} \int_0^1 \sum_{i \in [n]} \frac{dt}{2t} \left(\partial_i f(X_t) \cdot 2t\right)^2$$

$$= \mathbb{E}\, 2 \int_0^1 |\nabla f(X_t)|^2 \cdot t \, dt.$$

To summarize:

$$\mathrm{Var}[f] = \mathbb{E}[M]_1$$

$$\mathbb{E}\, d[M]_t = 2dt \cdot t \cdot \mathbb{E}|\nabla f(\vec{X}(t))|^2.$$

Thm (Talagrand '94):

$$\mathrm{Var}[f] \le C\,\mathbb{E}|\nabla f| \quad \left(= C\,\mathbb{E}\sqrt{h_f}\right).$$

Fix $c>0$.

$$T := \inf\{t \,;\, |\nabla f(X(t))| > c\} \wedge 1$$

Lemma A:

$$\mathbb{P}(T<1) > c\,\mathrm{Var}[f].$$

Lemma $\Rightarrow$ Thm:

$\vec{X}(t)$ is a martingale with iid coords.

$\Rightarrow$ $\nabla f(\vec{X}(t))$ is also a martingale.

$|\cdot|$ is convex.

jensen $\Rightarrow$ $\mathbb{E}\left[|\nabla f(X(1))| \,\middle|\, X_\tau \right] \geq$

$$\geq |\nabla f(X(\tau))|.$$

$$\underbrace{\mathbb{E}|\nabla f(X(1))|}_{\substack{\parallel \\ \text{RHS of } (*)}} \geq \mathbb{E}\left[\mathbb{1}_{\tau<1} \overbrace{|\nabla f(X(\tau))|}^{\substack{c \\ \wedge\!\shortmid}} \right]$$

$$\geq c\,\mathbb{P}(\tau<1) \overset{\text{Lemma}}{\geq} c \cdot c' \cdot \mathrm{Var}[f].$$

Fix $c > 0$.

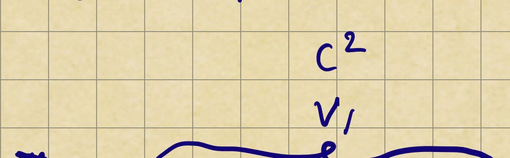
$\tau := \inf\{t \,;\, |\nabla f(X(t))| > c\} \wedge 1$

$\mathbb{E}\int t\, |\nabla f(X(t))|^2 dt = \mathrm{Var}[f$

Lemma A:

$\mathbb{P}(\tau<1) > c\ \mathrm{Var}[f].$

First assume $\mathrm{Var}[f] \geq \frac{1}{100}$.



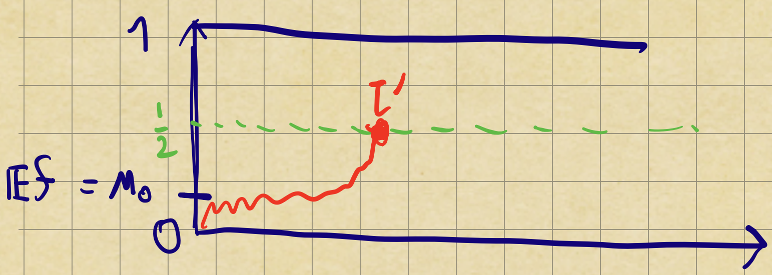
$$\mathbb{E}[M]_\tau = \mathbb{E}\int_0^\tau t\,|\nabla f(X(t))|^2\,dt$$

$$\frac{1}{100} \leq \mathrm{Var}[f] = \mathbb{E}[M]_1 = \underbrace{\mathbb{E}[M]_\tau}_{\substack{\wedge\!\wedge \\ c^2}} + \underbrace{\mathbb{E}\big([M]_1 - [M]_\tau\big)}_{\substack{\| \\ \mathbb{P}(\tau\leq 1)\cdot \underbrace{\mathbb{E}[\mathrm{Var}(M_1 - M_\tau)\,|\,\tau<1]}}}$$

$$\leq c^2 + \mathbb{P}(\tau \leq 1)$$

Take $c = \frac{1}{100}$

$$\Rightarrow \mathbb{P}(\tau\leq 1) \geq \frac{1}{200}.$$



$$\mathbb{P}(\exists t \text{ s.t } M_t \geq \tfrac{1}{2}) \geq \mathbb{E}[f] \geq c\,\mathrm{Var}[f].$$

"dt" a small number.

$t \in \mathbb{Z}\,dt = \Lambda$

For Z_t, denote $dZ_t = Z_{t+dt} - Z_t$.

Define $(B_t)_{t\in\mathbb{R}}$ so that $dB_t \sim N(0, dt)$

$(dB_t)_{t\in\Lambda}$ iid.

Brownian Motion.

$(B_{t_1} - B_{t_2}) \sim N(0, t_2 - t_1)$.

Let $\vec{B}(t)$ be s.t. $B(t)_i$ is a B-M.

Define $\mathcal{F}_t$ to be σ-algebra generated by $(B(s))_{0 \le s \le t}$.

Suppose that $\sigma(t)$ is a matrix-valued process s.t $\sigma(t)$ is $\mathcal{F}_t$-measurable.

Define $X(t)$ by

$$dX(t) = \underbrace{\sigma_t \, dB(t)}_{\sim N(0, \sigma_t^T \sigma_t \, dt)}.$$

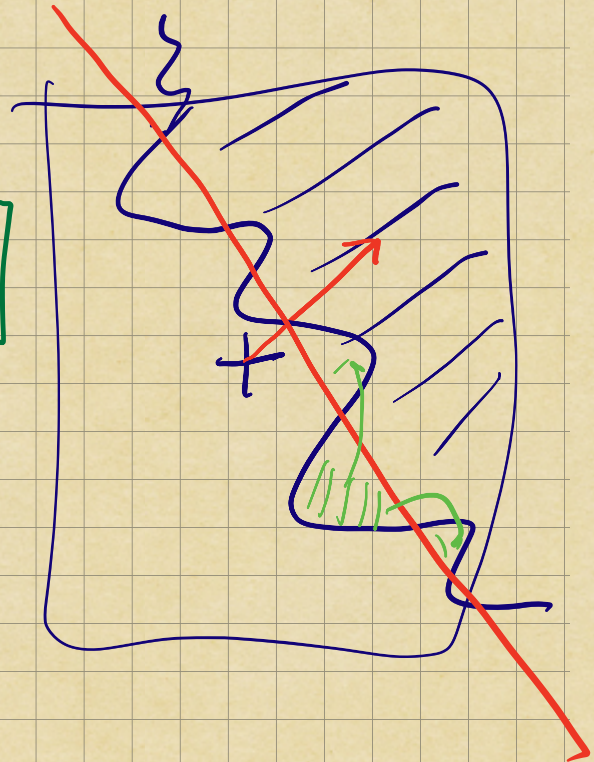
X_t is going to be a martingale.

Choose $(\sigma_t)_{ii} = \sqrt{(1 + X_i(t))(1 - X_i(t))}$; $(\sigma_t)_{ij} = 0 \quad i \neq j$

$$d X_i(t) = \sqrt{(1 + X_i(t))(1 - X_i(t))}$$

We'll use this process to prove Maj. is Stablest [MOO '08].

[joint Mikulincer - Raghavendra].



$$M_t = f(X_t)$$

$$dM_t = \nabla f(X_t) \cdot dX_t \quad [+\, o(dt)]$$

$$= \nabla f(X_t)\, \sigma_t \, dB_t$$

$$\mathrm{Var}(M_t) = \mathbb{E}\int_0^t (dM_t)^2 = \mathbb{E}[M]_t .$$

$$d[M]_t = (dM_t)^2 = (\nabla f(X_t)\, \sigma_t \, dB_t)^2$$

$$\text{"="} \quad |\sigma_t \nabla f(X_t)|^2 \, dt .$$

$$\mathrm{Var}(M_t) = \mathbb{E}\int_0^t |\sigma_t \nabla f(X_t)|^2 \, dt \qquad \boxed{\frac{\int x f(x)}{\int f} \leq \frac{I(\int f)}{\int f}} .$$

<u>Thm</u> (level-1 inequality): $\quad \dfrac{\nabla f(0)}{\int f} = \dfrac{\int x f(x)\, dx}{\int f}.$

For every bool. fn f, $\forall\, x \in [-1,1]^n$, we have

$$|\sigma_t \nabla f(x)| \leq I(f(x)) + \Psi(\|\nabla f(x)\|_\infty)$$

where $I(s) = \Phi'(\Phi^{-1}(s))$ where Φ gaussian CDF.

Ψ is some fn s.t $\lim_{s\to 0} \varphi(s) = 0$.

If f is the majority fn,

$$|\sigma_t \nabla f(x)| = I(f(x)) + O(\Psi(\|\nabla f(x)\|_\infty)).$$

Maj is stablest:

$\underbrace{Var[f(X_t)]}$ maximized by majority up to

$\|$ $\max_i I_i(f)$.

$Stab_{\rho(t)}[f]$.

We're given a bool. fn f. f is associated with a martingale M_t. Suppose $g(x) = maj(x)$
Let N_t be the martingale corresp. to g.

$\alpha_t = \|\nabla f(X(t))\|_\infty$

$$d[M]_t = |\sigma_t f(X_t)|^2 dt \leq I(M_t) + \alpha_t$$

$$d[N]_t = |\sigma_t g(X_t)|^2 = I(N_t) - \beta_t$$

Lemma 1: $\max_i I_i(f) = o(1)$ then $\max_{s \le t} \alpha_s = O_t(1)$.

Lemma 2: Suppose that $(M_t)_t$, $(N_t)_t$ are martingales s.t.

$$\left.\begin{array}{l} d[M]_t \le I(M_t) \\ d[N]_t = I(N_t) \end{array}\right\} \text{ for all } t \text{ a.s.}$$

$\Rightarrow \mathrm{Var}(N_t) \geq \mathrm{Var}(M_t) \quad \forall t.$

$\Rightarrow \quad \mathrm{Stab}_{\rho(t)}[f] \le \mathrm{Stab}_{\rho(t)}(\mathrm{maj}) + \text{correction}$

correction $\searrow 0$ when $\max_i I_i(f) \searrow 0$.

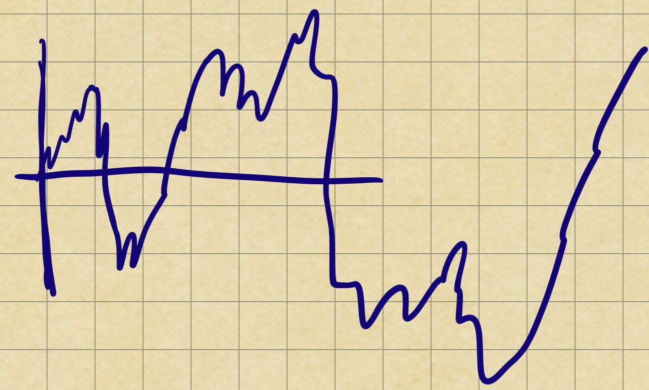
Dambis / Dubins-Schwartz thm:

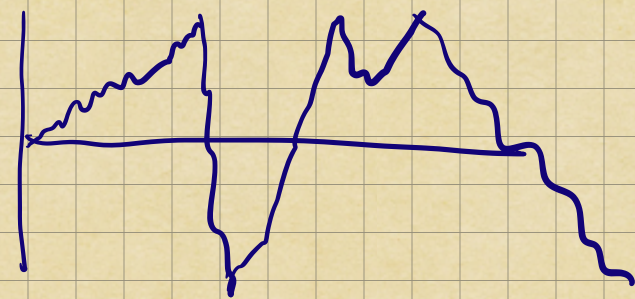
if M_t is a continuous martingale the $\exists$ W_s Brownian motion s.t

$$M_t = W_{[M]_t}$$

$M_t = W_{[M]_t}$

$N_t = W_{[N]_t}$





$X_1 \in \{-1, 1\}$.

$\mathbb{E}\, M_t^2$

$\mathbb{E}\, M_t^q$