Global hypercontractivity inequality on ε-product spaces

Tom Gur

Noam Lifshitz

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joint work with

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(2) $\exists \rho_0 > 0$, **s.t.** $\forall 0 < \rho < \rho_0, \|T_\rho f\|_4^4 \le C\|f\|_2^4$

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* where $\|f\|_p^p = \mathbb{E}_\mu[f^p]$

Notes

T_ρ **is a semigroup operator defined as**

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Over Gaussian space T_ρ is the Ornstein-Uhlenbeck semigroup.

Over the Boolean hypercube T_ρ is the noise operator.

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$$\|T_\rho f\|_4^4 \le C\|f\|_2^4 \Rightarrow \|T_\rho f\|_q^q \le C_{p,q}\|f\|_p^q \;\forall 1 < p < q$$

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$$||T_\rho f||_4^4 \leq C||f||_2^4 \Rightarrow ||T_\rho f||_q^q \leq C_{p,q}||f||_p^q \;\forall 1 < p < q$$

Sometimes written as $||f||_4^4 \leq C_d||f||_2^4$ **rather than**

$$||T_\rho f||_4^4 \leq C||f||_2^4$$

Hypercontractivity inequality

Theorem: We say (Ω, μ) is hypercontractive if there exists C such that $\forall f \in L^2(\mu)$ $\|f\|_4^4 \leq C(\deg(f)) \cdot \|f\|_2^4$

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Implications

Improved (anti-)concentration for f:

$\forall t > 0$

$$\Pr[\,|f| \geq t\|f\|_2] \leq C/t^4$$

$\forall t \in (0,1)$

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By Markov's

By Paley-Zygmund

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Level-d inequality: There exists C such that for all $f : \Omega \to \{0,1\}$

$$\|f^{\leq d}\|_2 \leq C^{1/4} \cdot \|f\|_2^{3/2}$$

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Hypercontractivity $\Rrightarrow$ **Weights of low density boolean functions concentrate on high degrees**

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For certain Markov chain G (defined by T_ρ) over (Ω, μ):

Small Set Expander (Qualitative): G is a small set expander if every small set of vertices has most adjacent edges outside the set.

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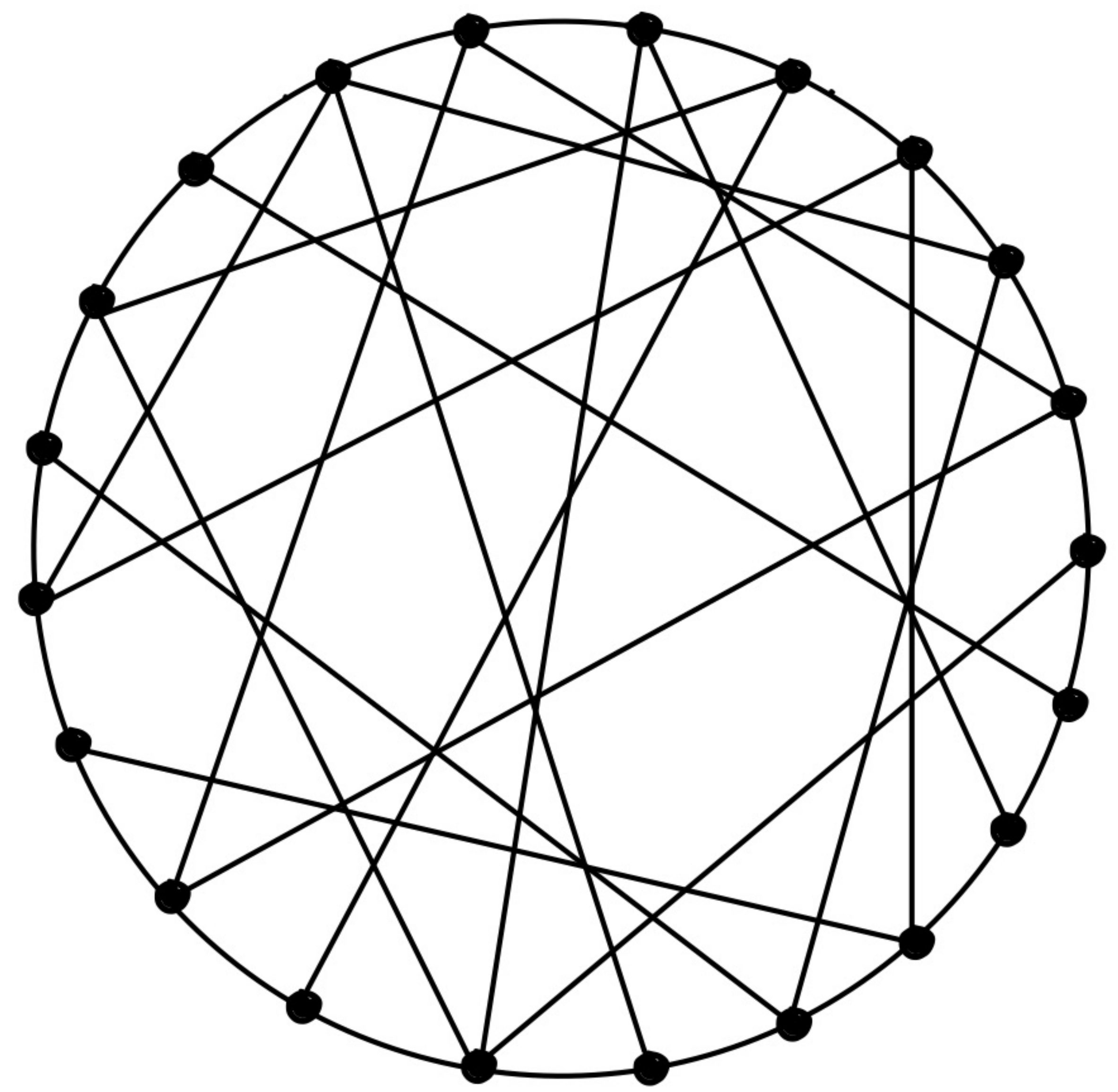

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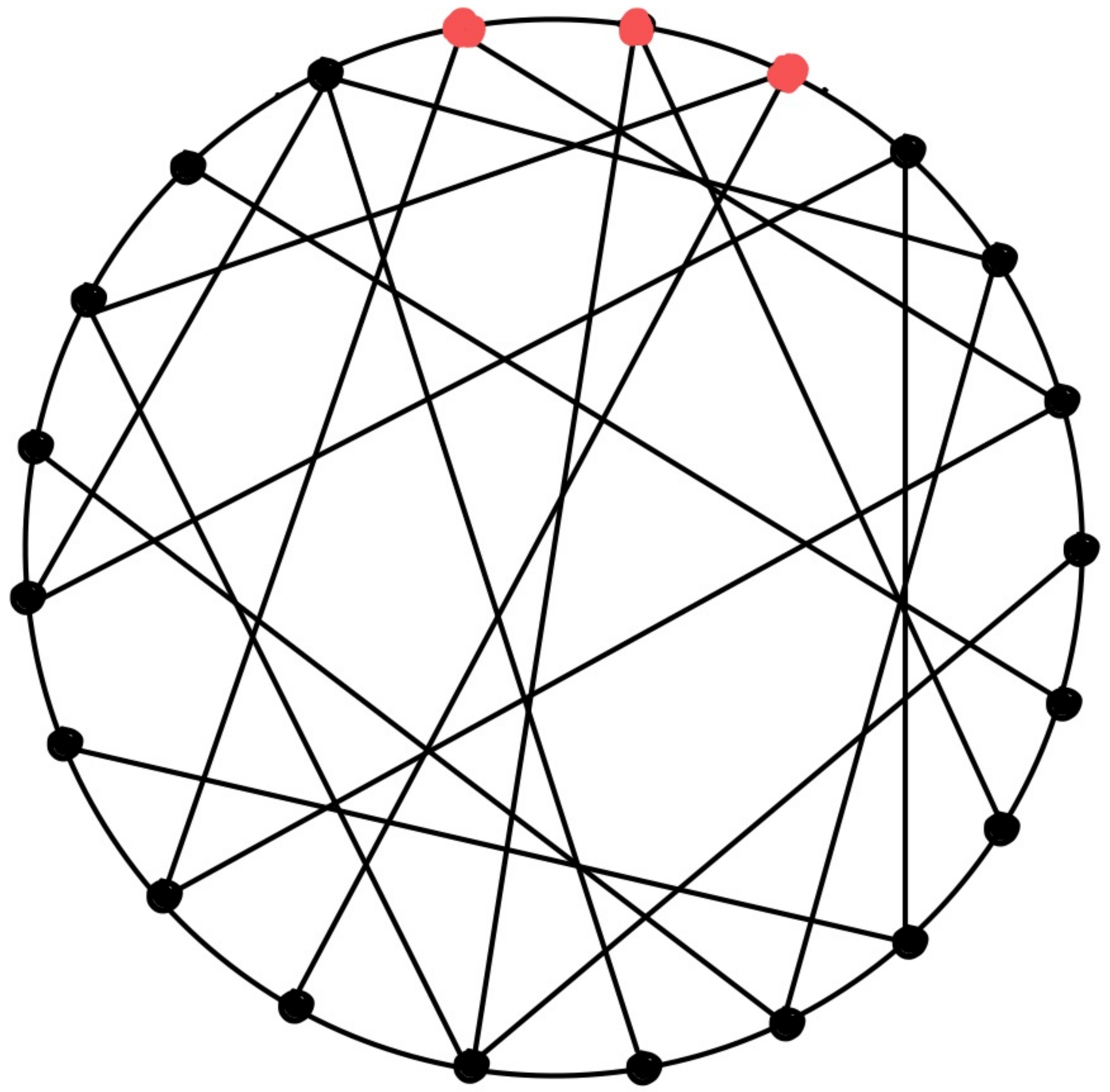

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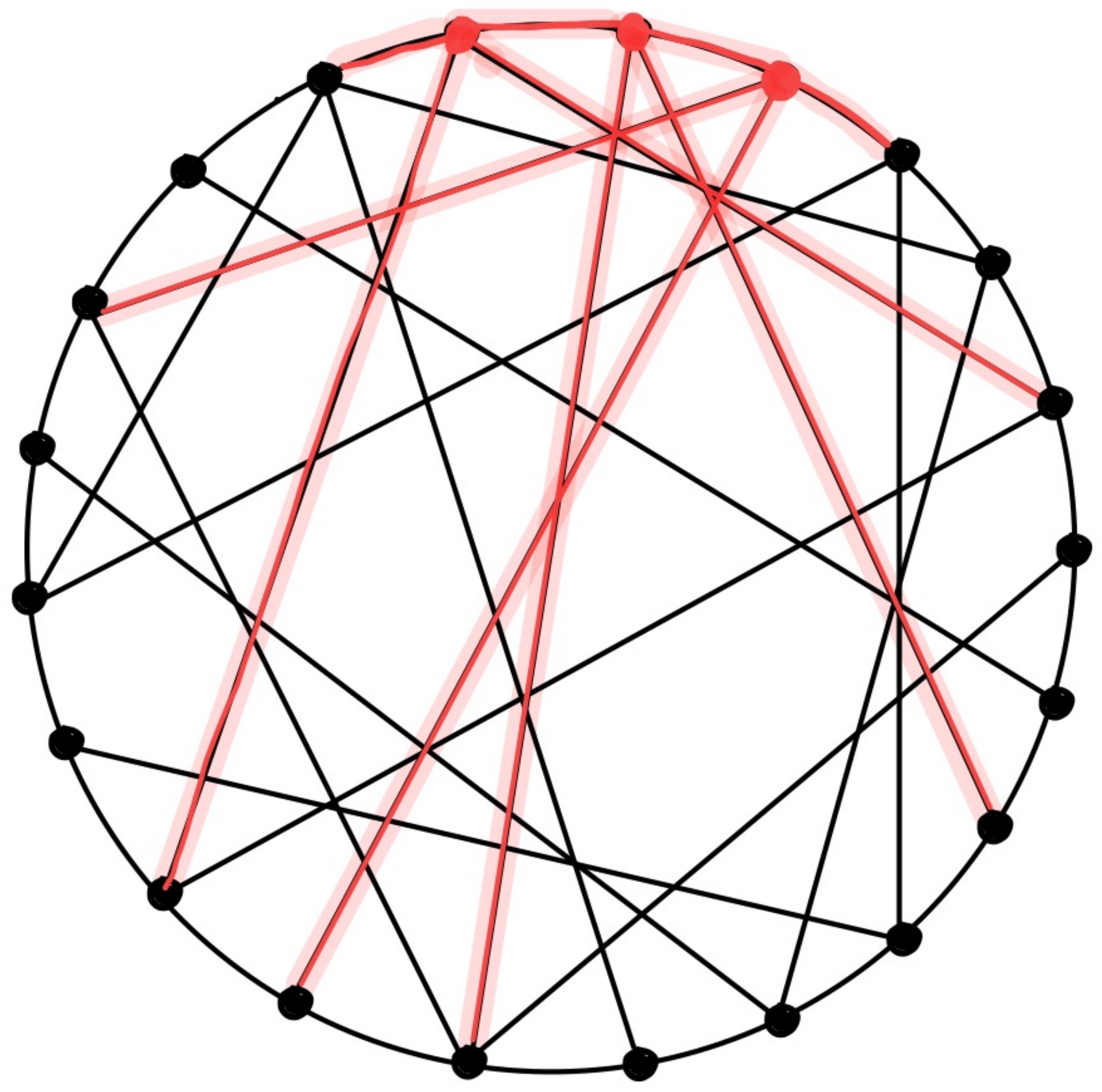

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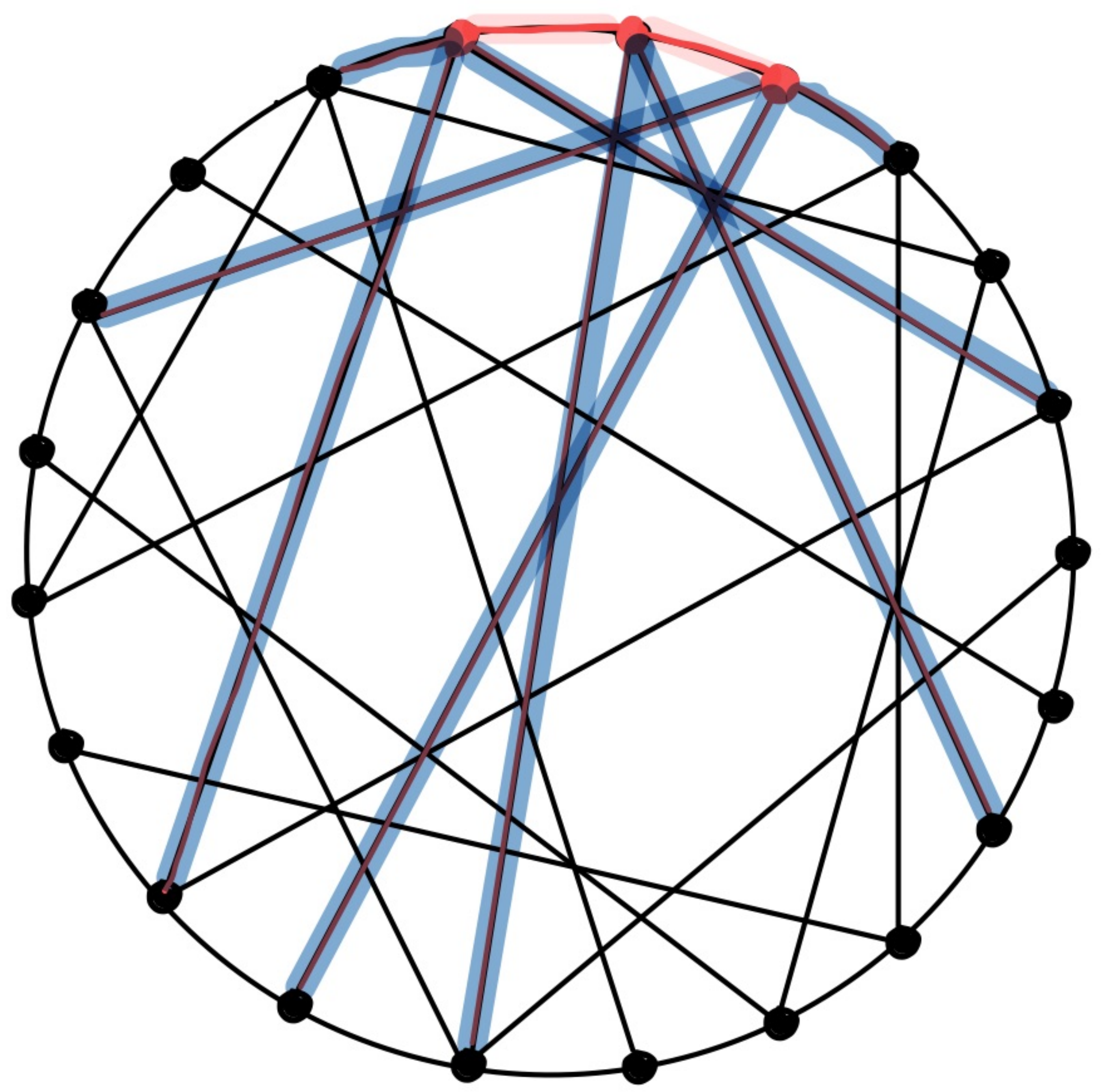

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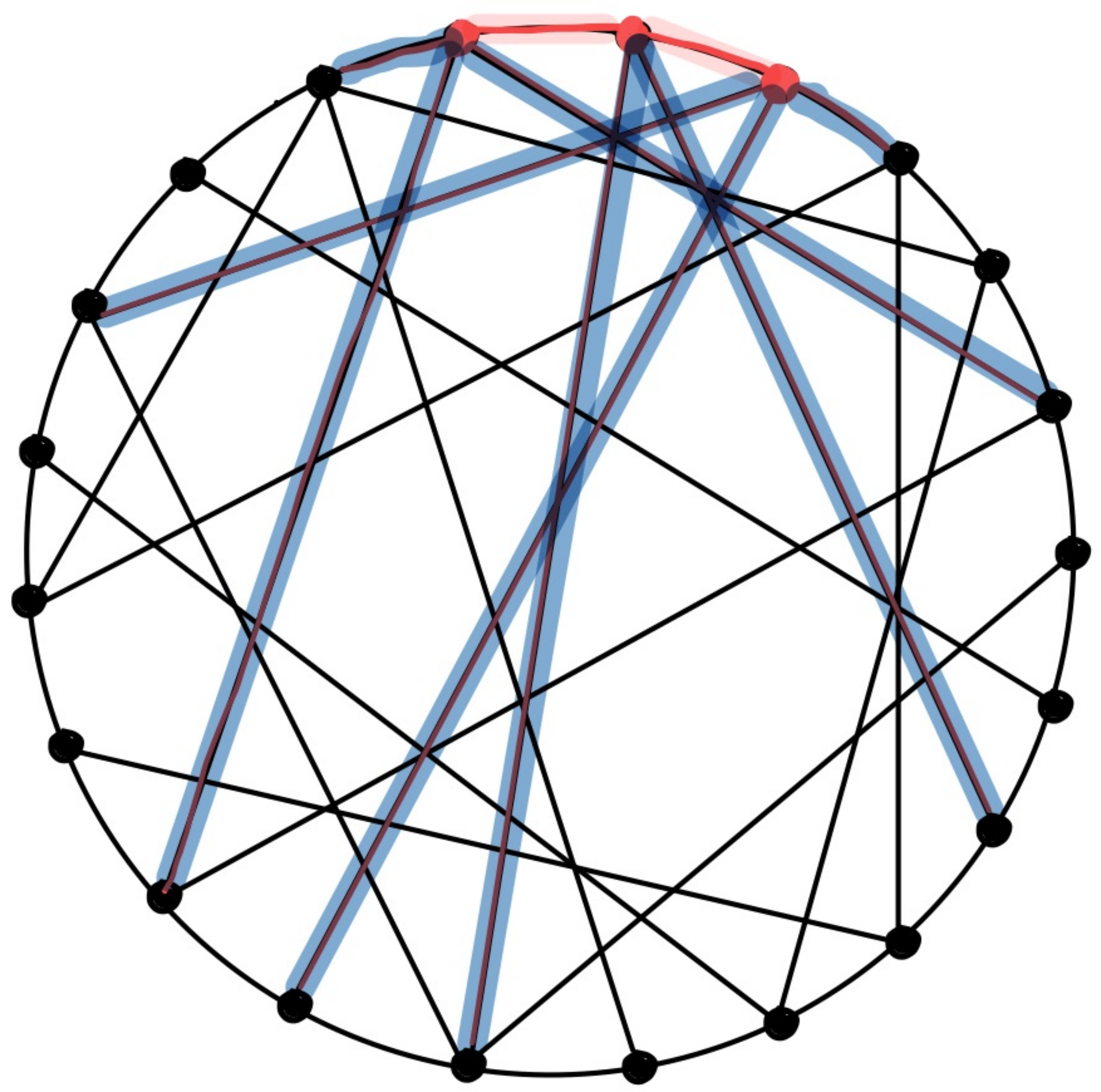
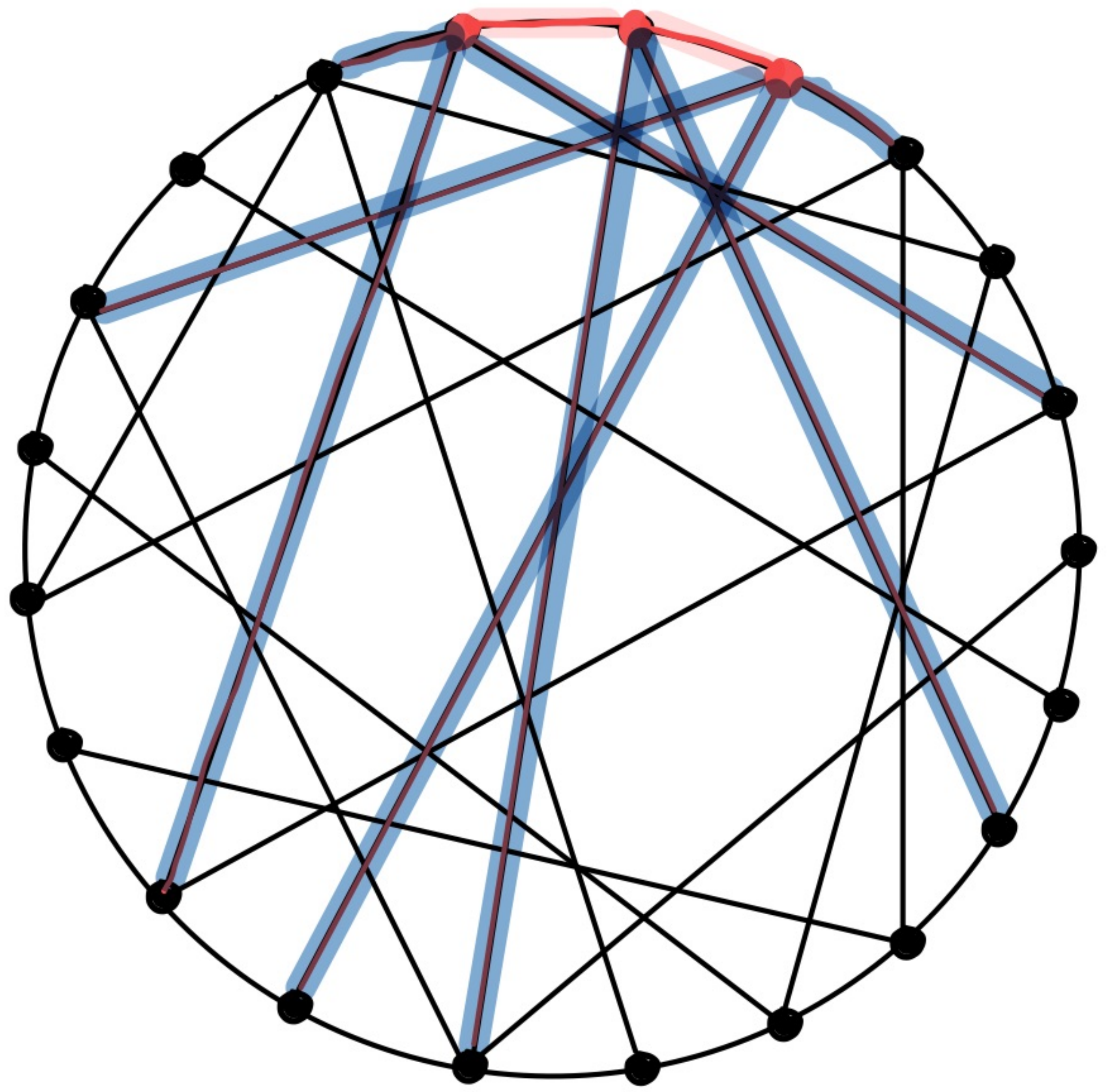
Hard instance for Unique Games: small set expanders with many large eigenvalues?



Agreement test on graphs: for Grassmann graph, 2-to-2 Games Conjecture

Hypercontractivity inequality



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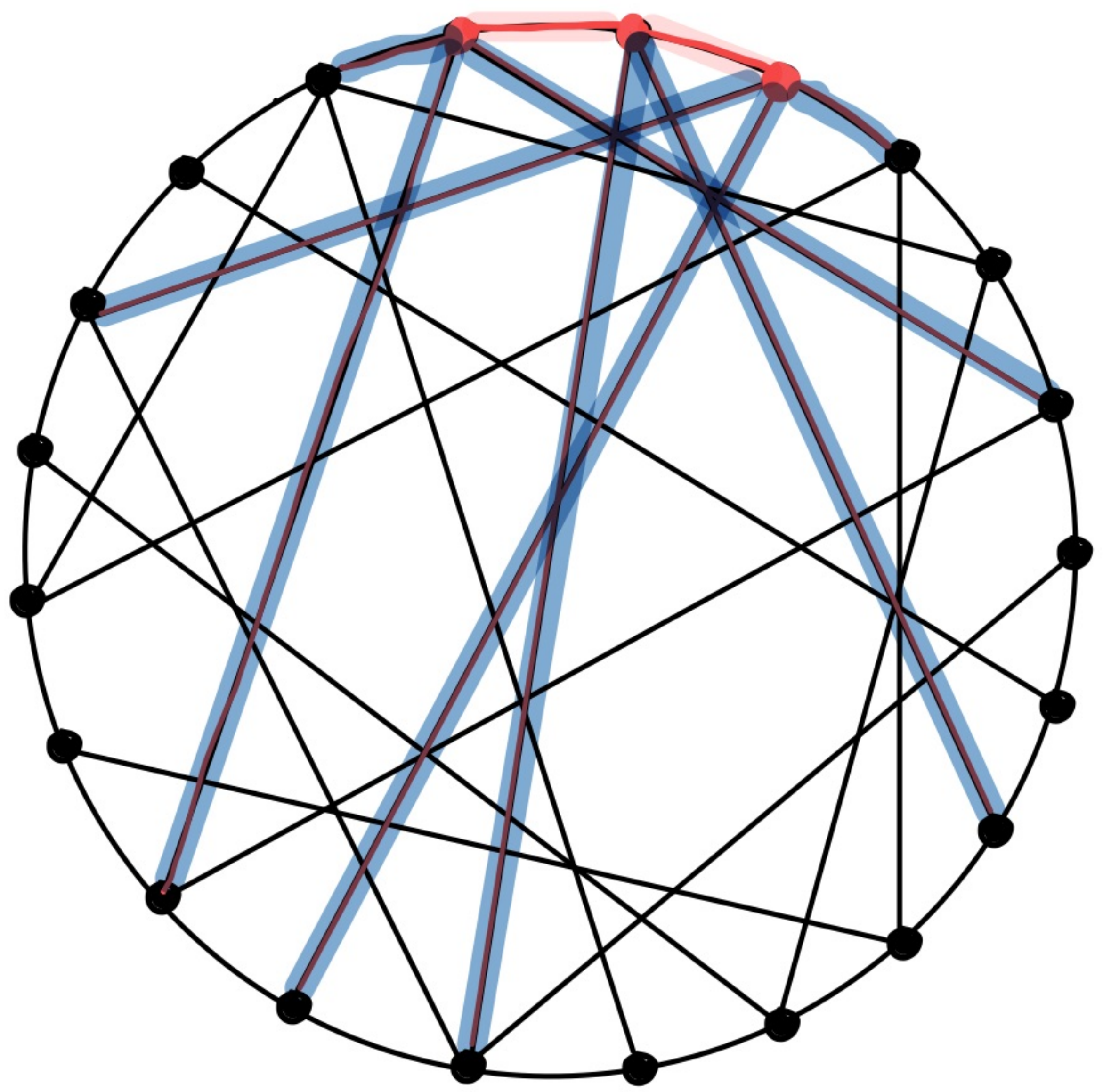

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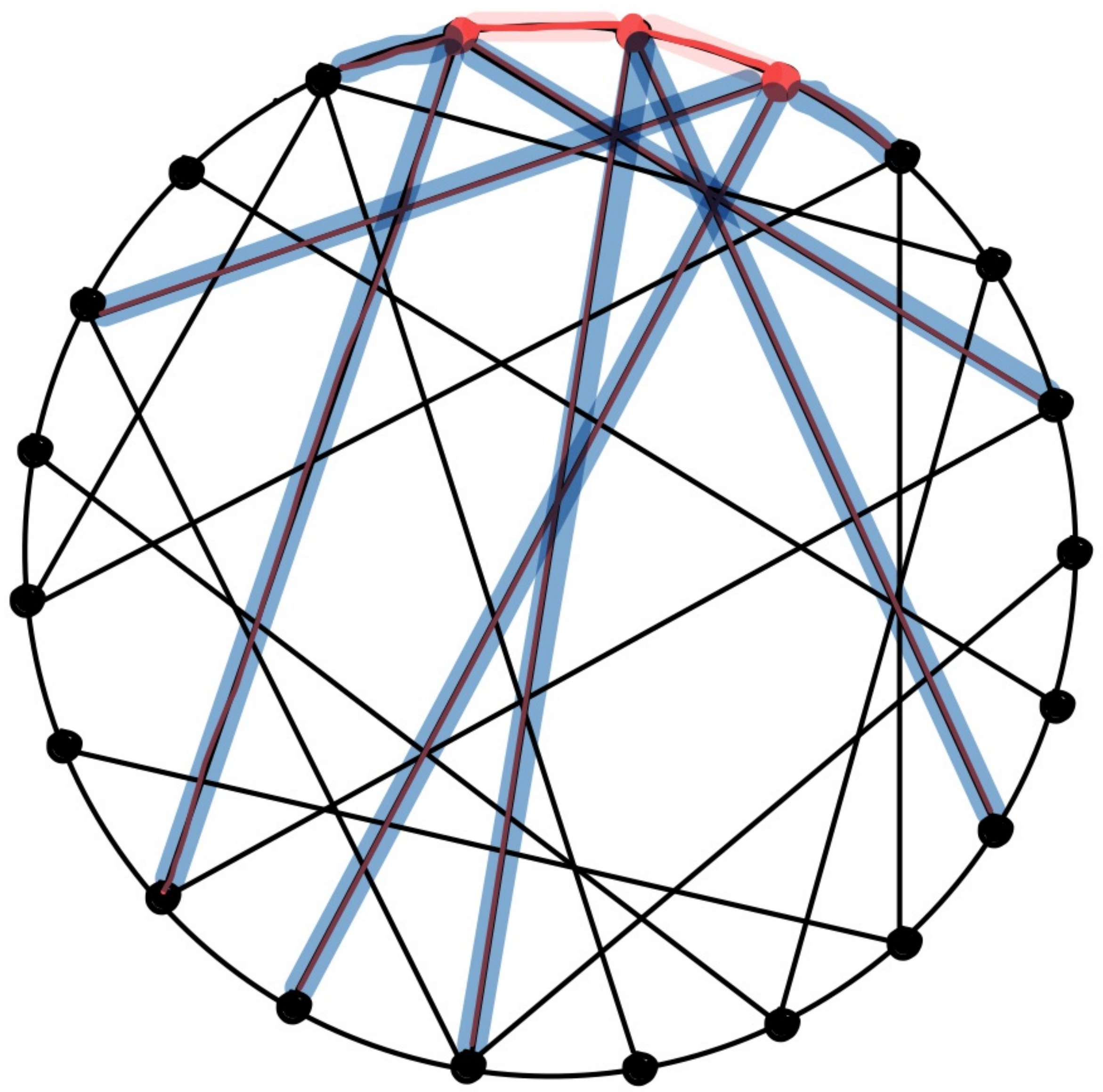

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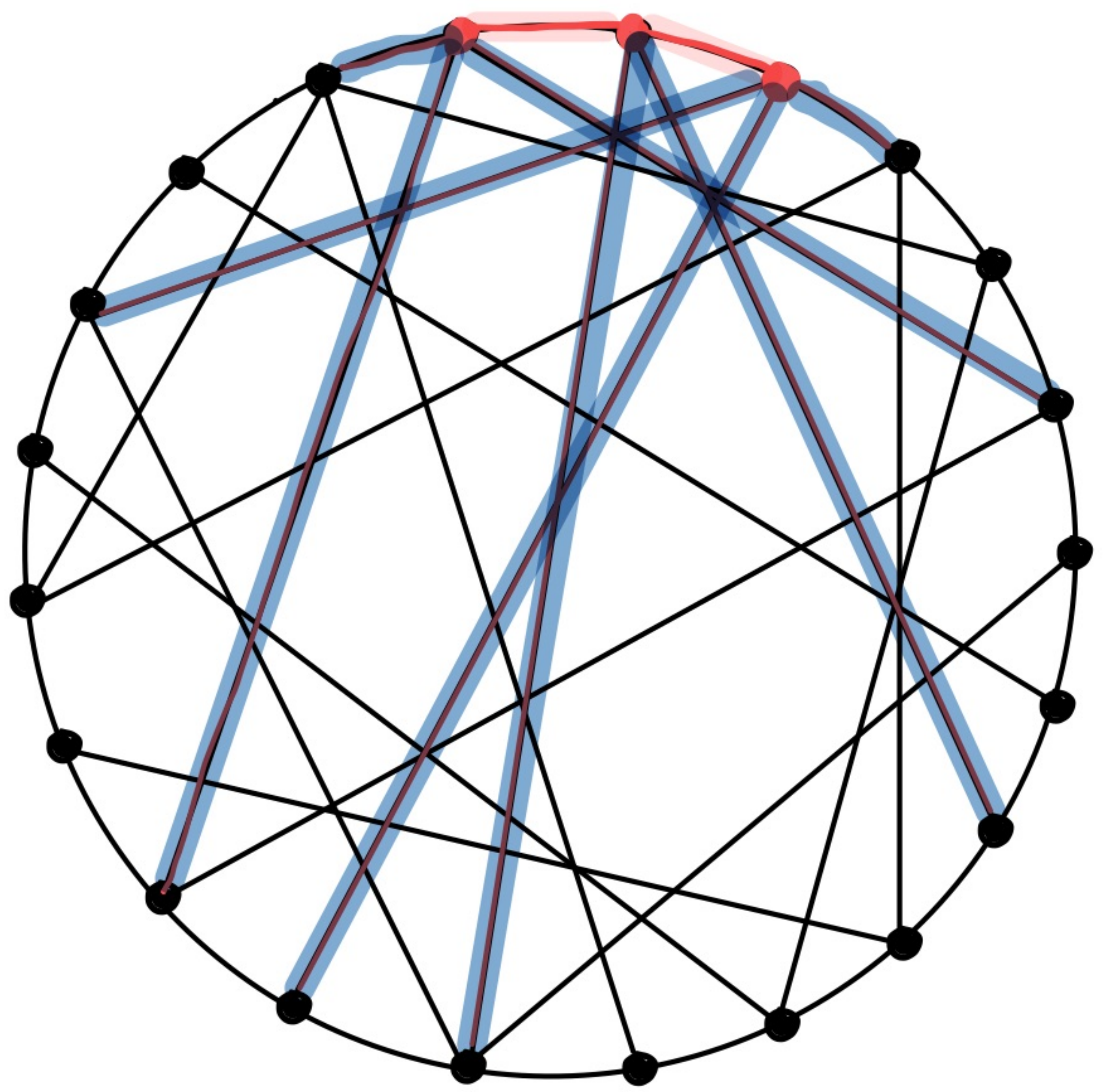

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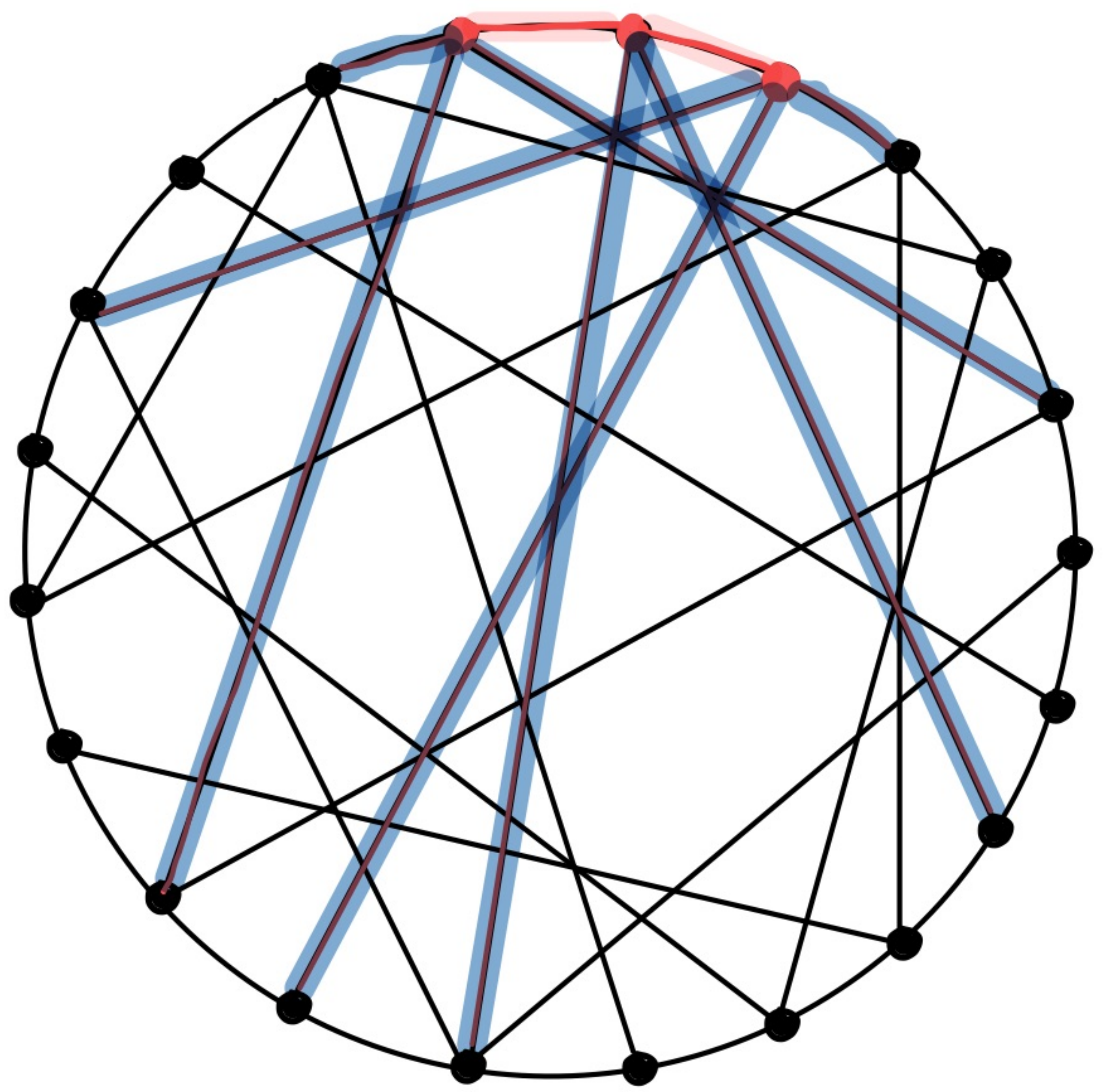

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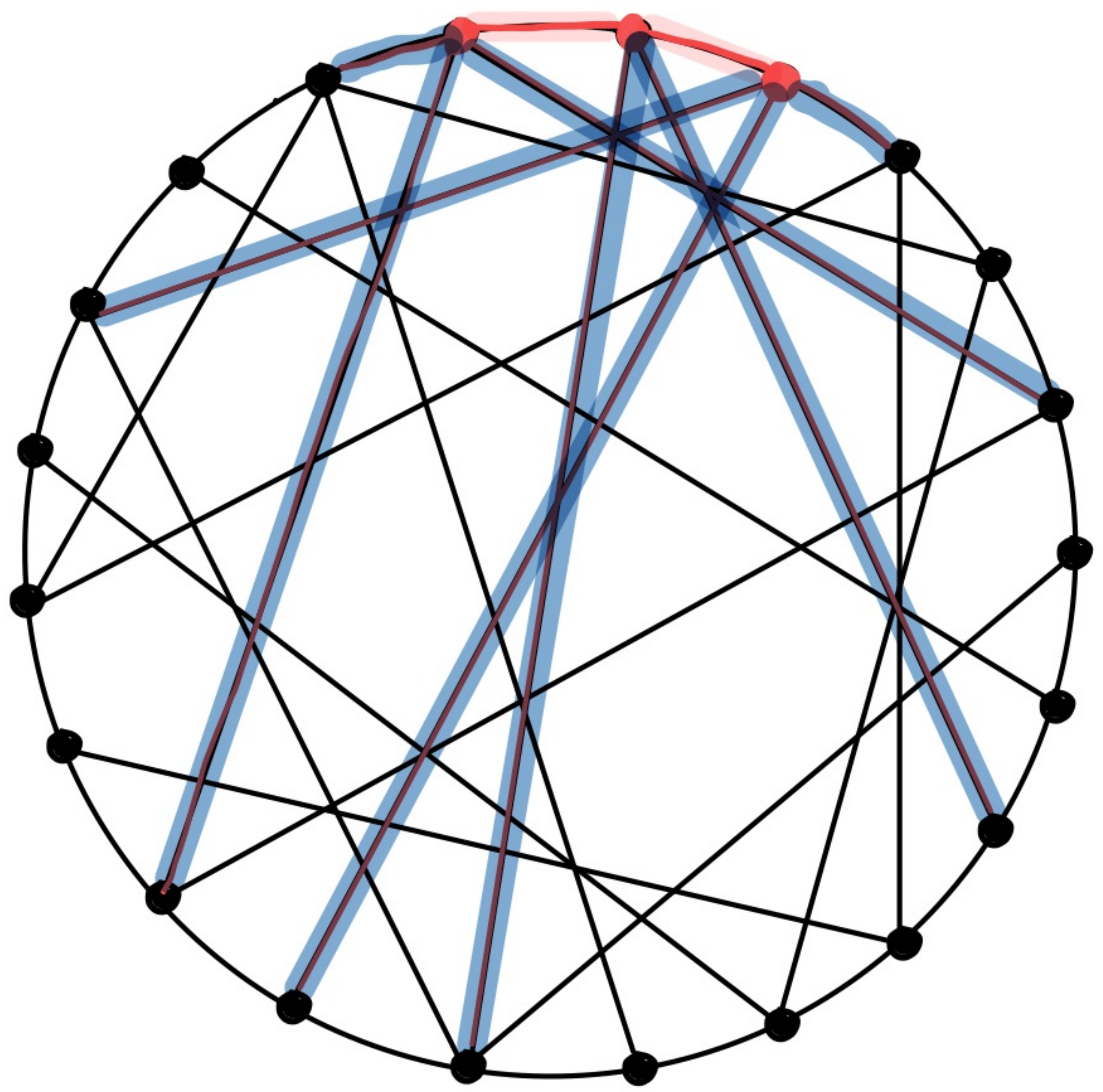

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$$\leq (C_d^{1/4} \|f\|_2^{1/2} + \rho^d)\|f\|_2$$

Hypercontractivity inequality example

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Bonami lemma: for $f : \{\pm 1\}^n \to \mathbb{R}$, $\|f\|_4^4 \leq 9^{\deg(f)} \cdot \|f\|_2^4$

More examples

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[Bon]	$\{\pm 1\}^n$, Unif	9^d	/
[KLLM]	general product space	$100^d \delta / \|f\|_2^2$	f is global
[FKLM]	S_n	$\exp(d^3)\delta / \|f\|_2^2$	f is global
[OW]	$\binom{[n]}{k}$, Unif	$\left(\frac{n^2}{k(n-k)}\right)^{O(n)}$	/
[FOW]	multi-slice, Unif	$\tilde{O}(n)^{2n}$	/

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Holds for more general almost product spaces?

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close to pair-wise independent

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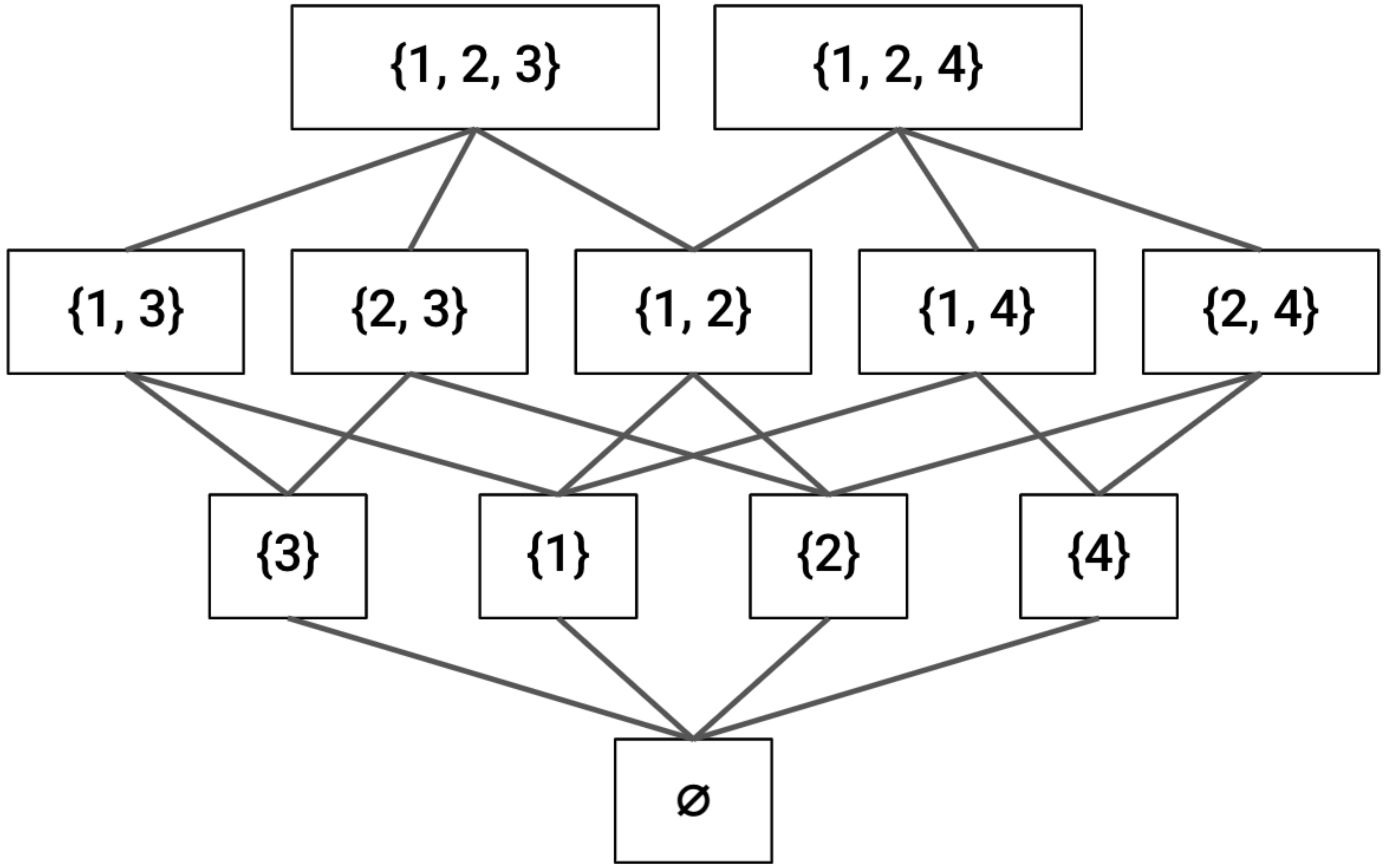
$(E = V^3, \mu)$



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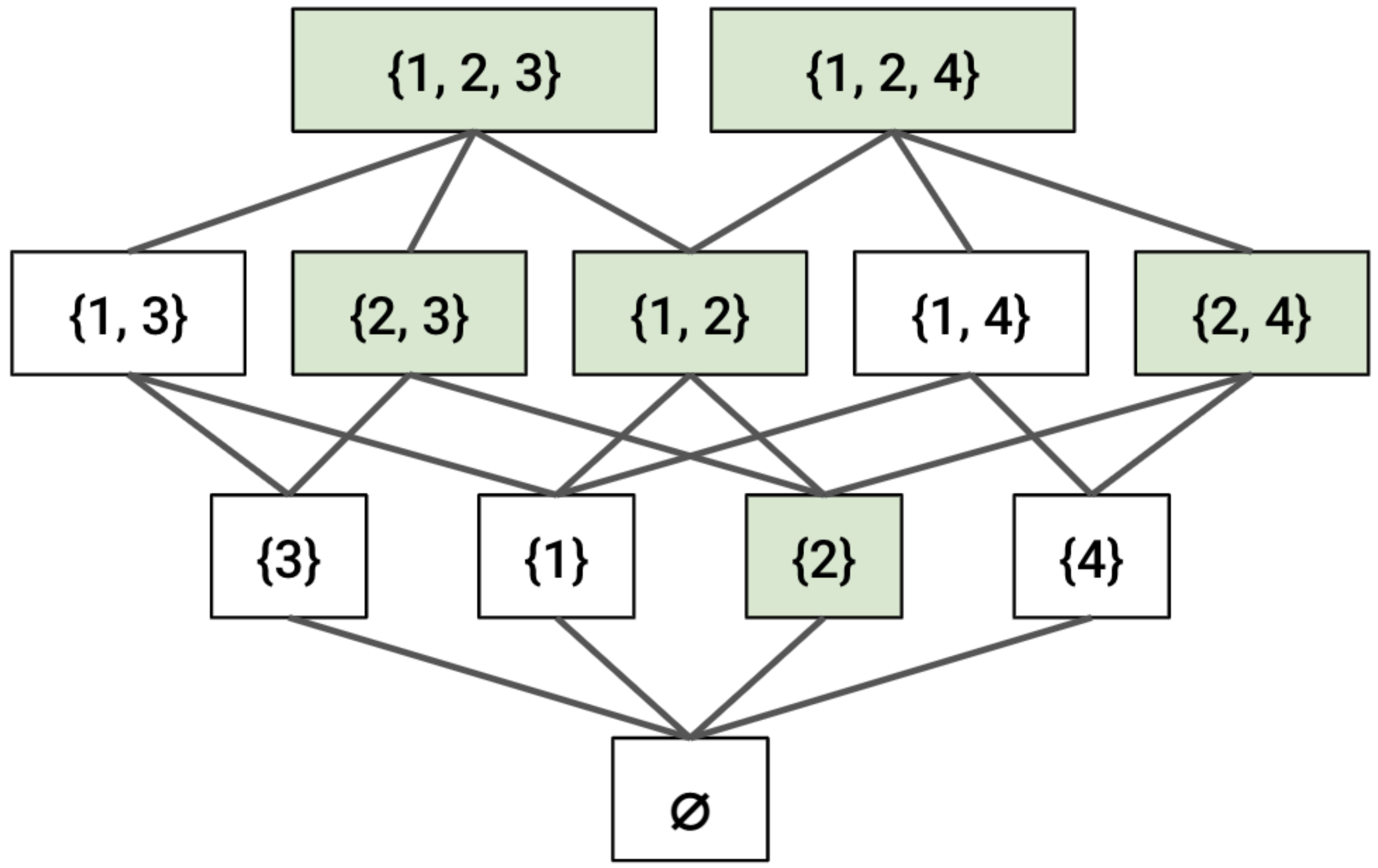

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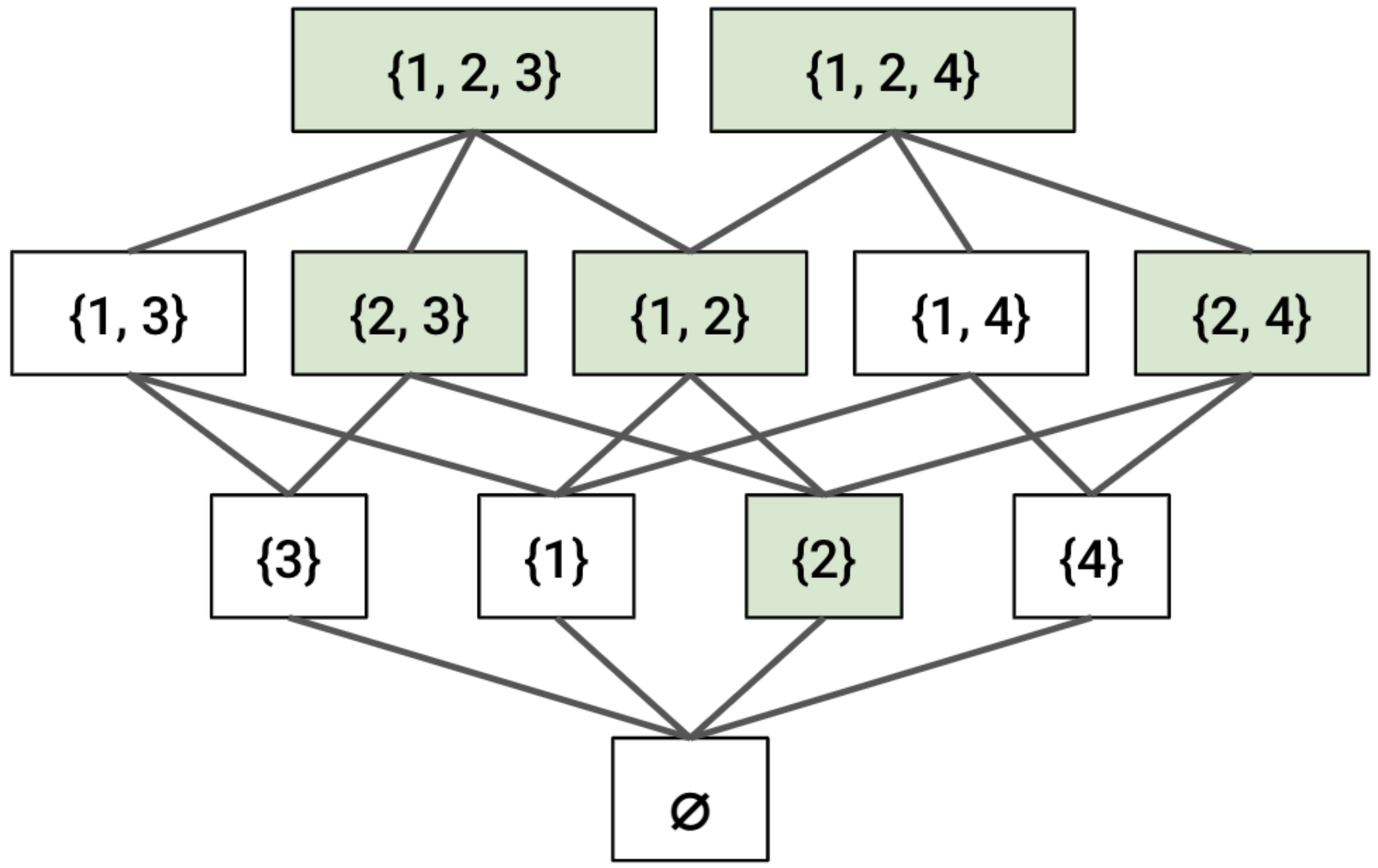

After removing $\{2\}$:

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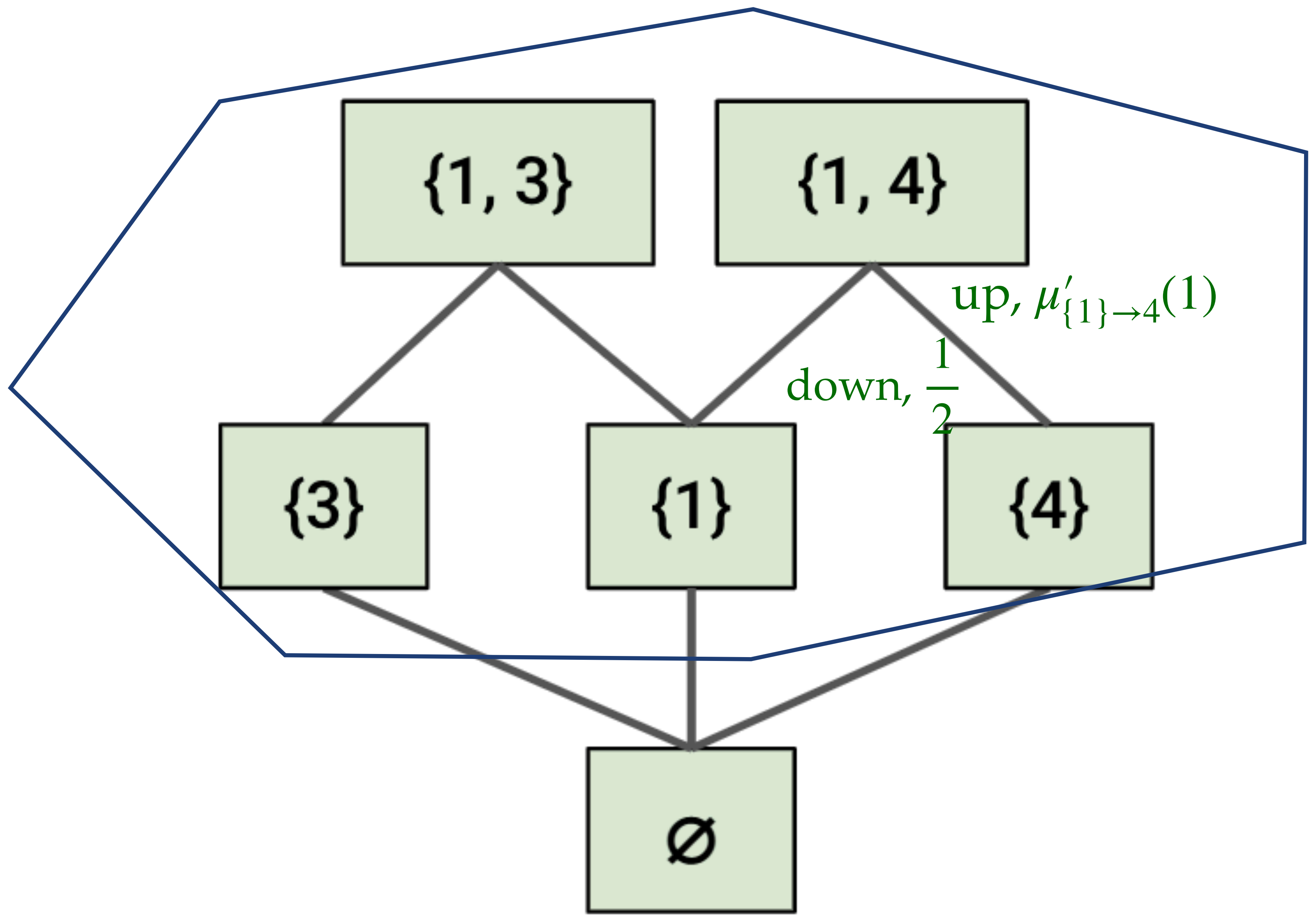

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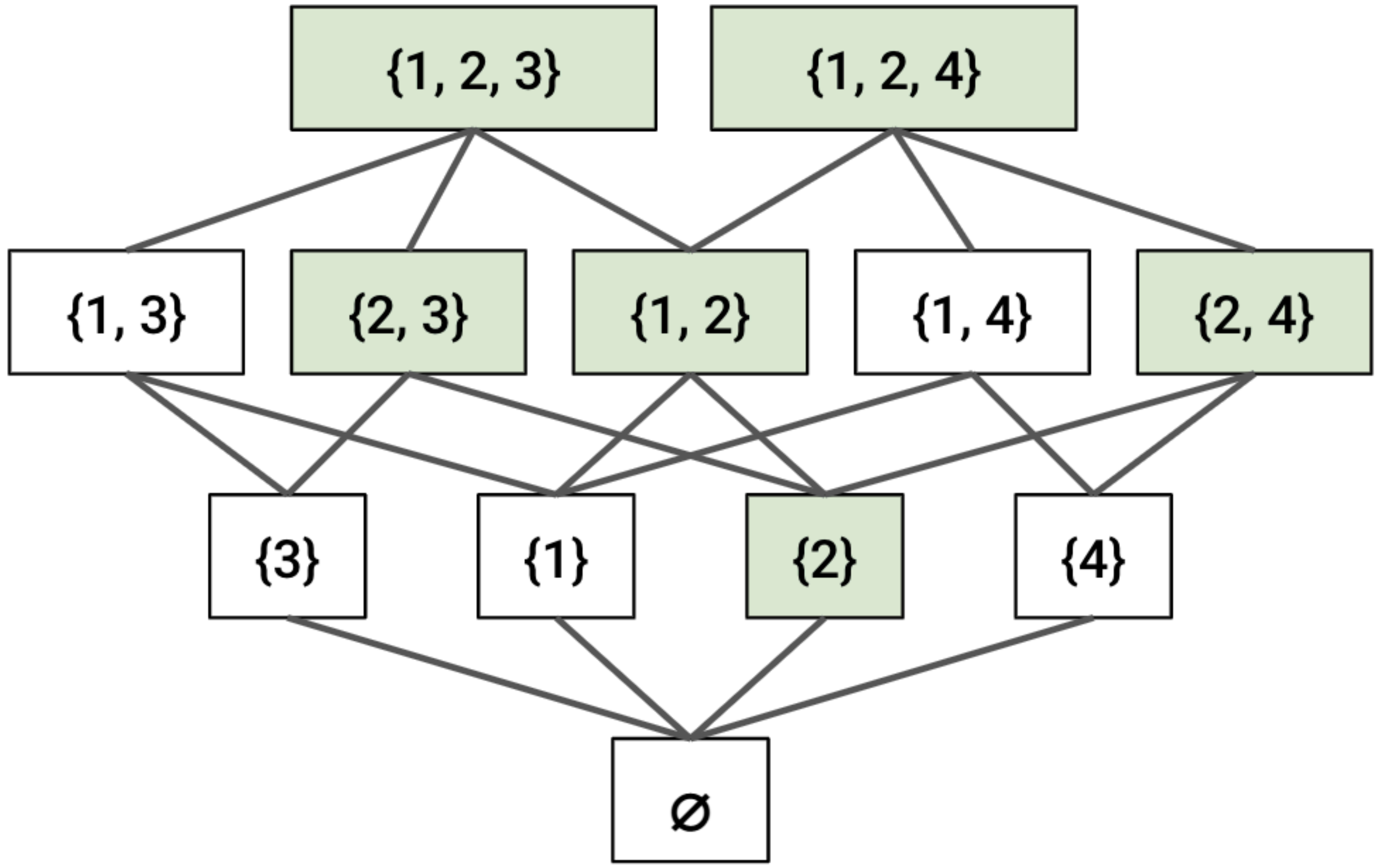

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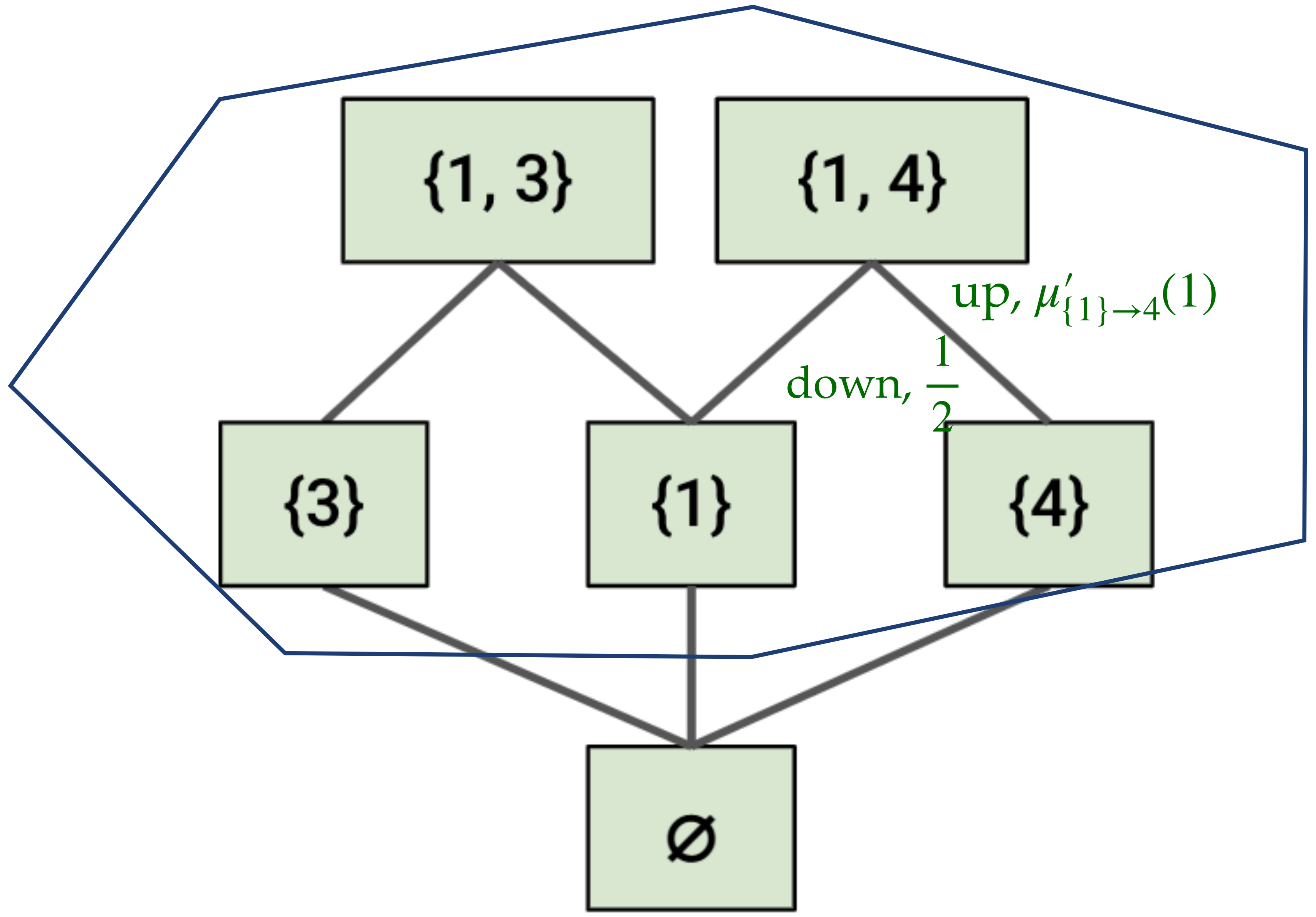

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(walk mixes fast)

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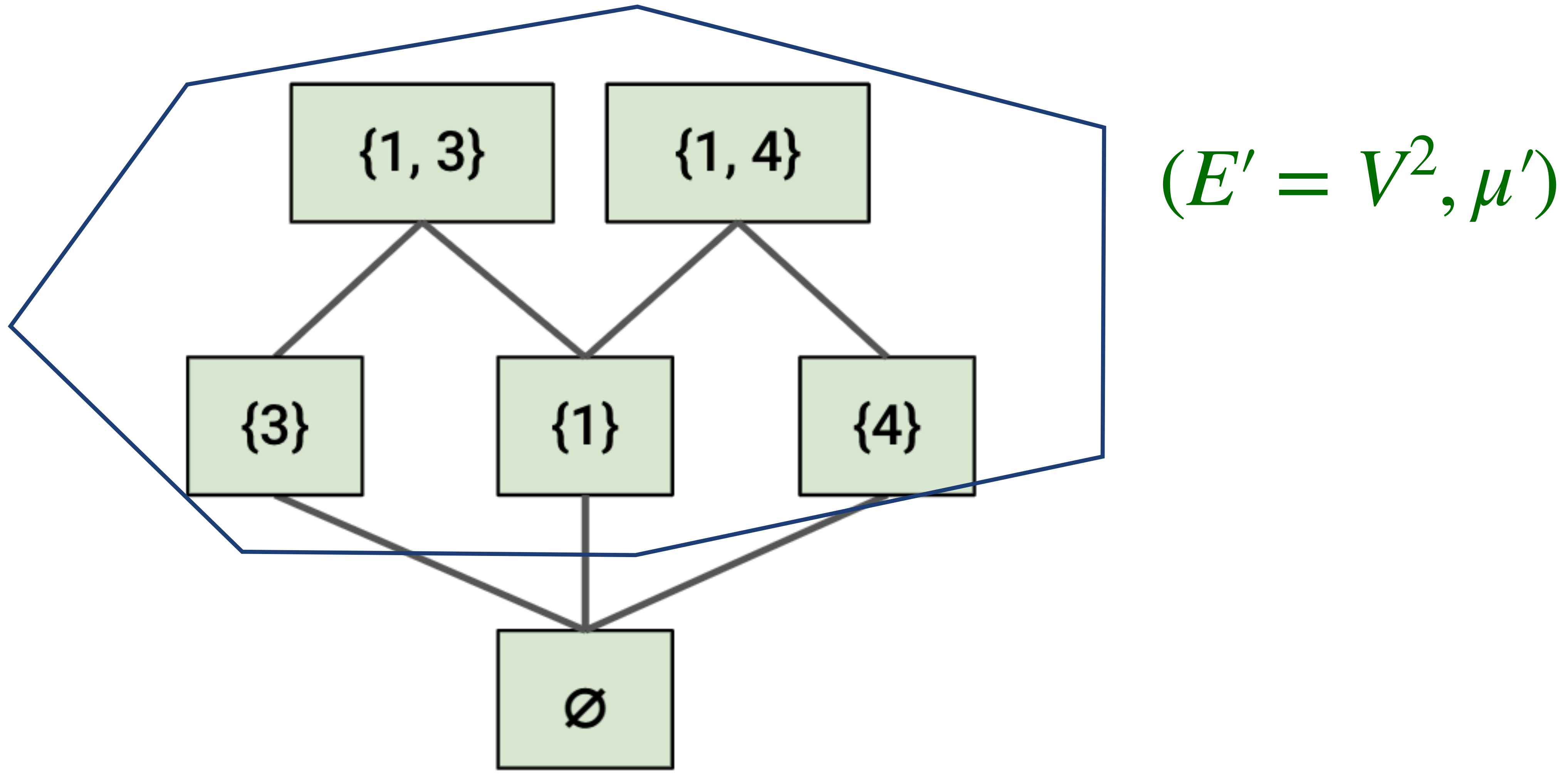

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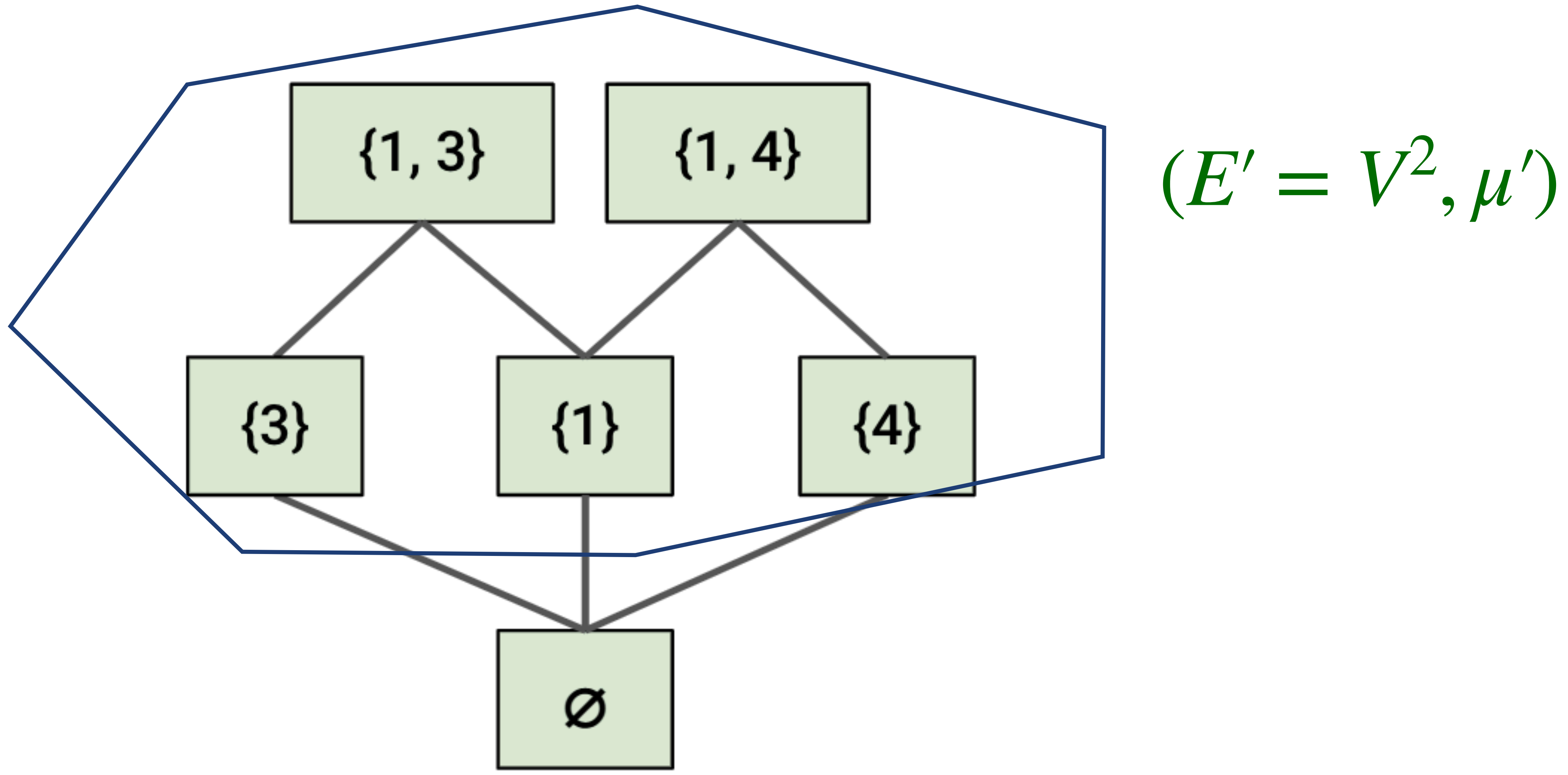
ϵ-product space: (V^k, μ) s.t.



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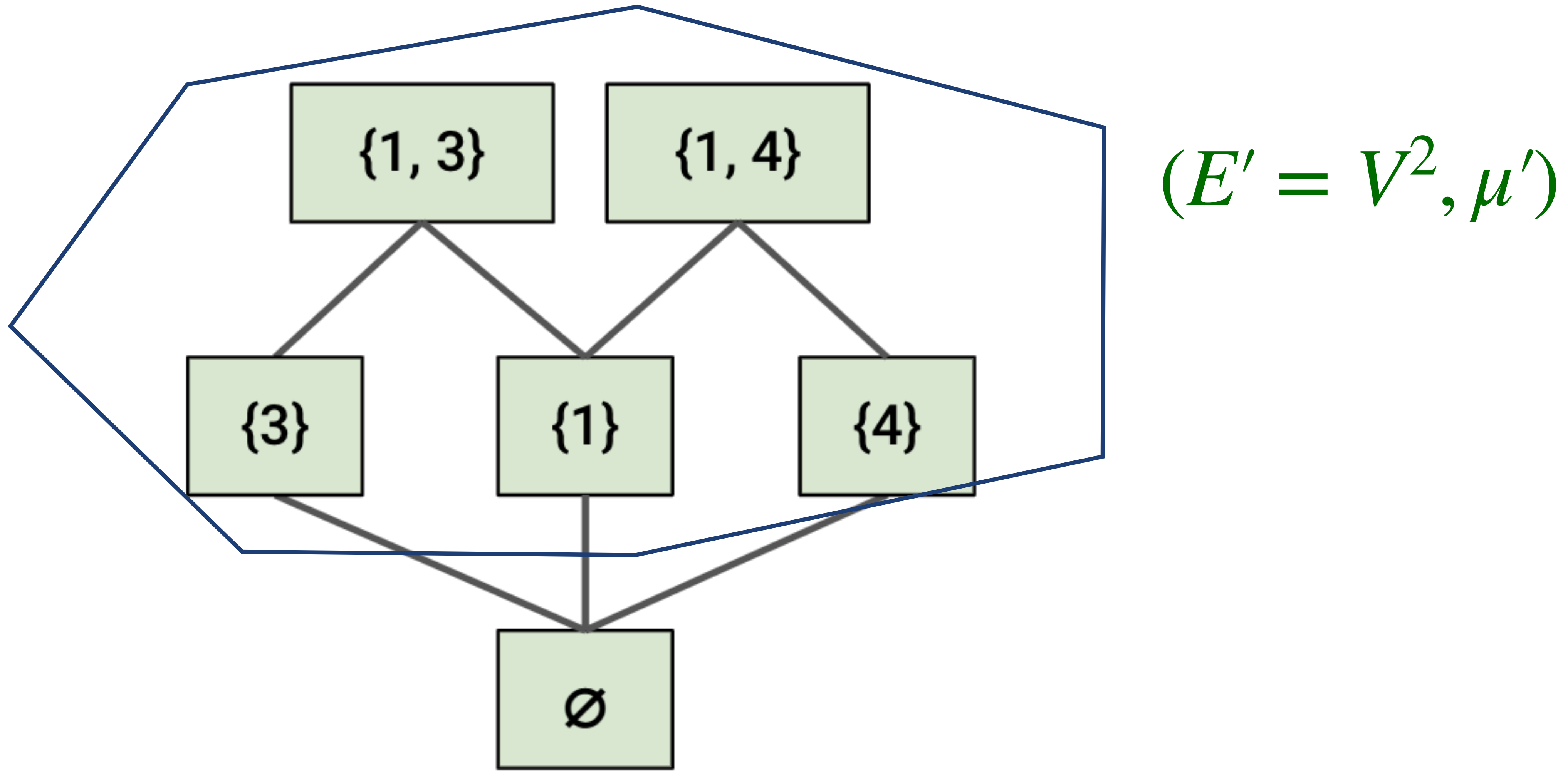

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variables in $V^{k-|S|}$ have identical marginal dist $\mu'^{(0)}$

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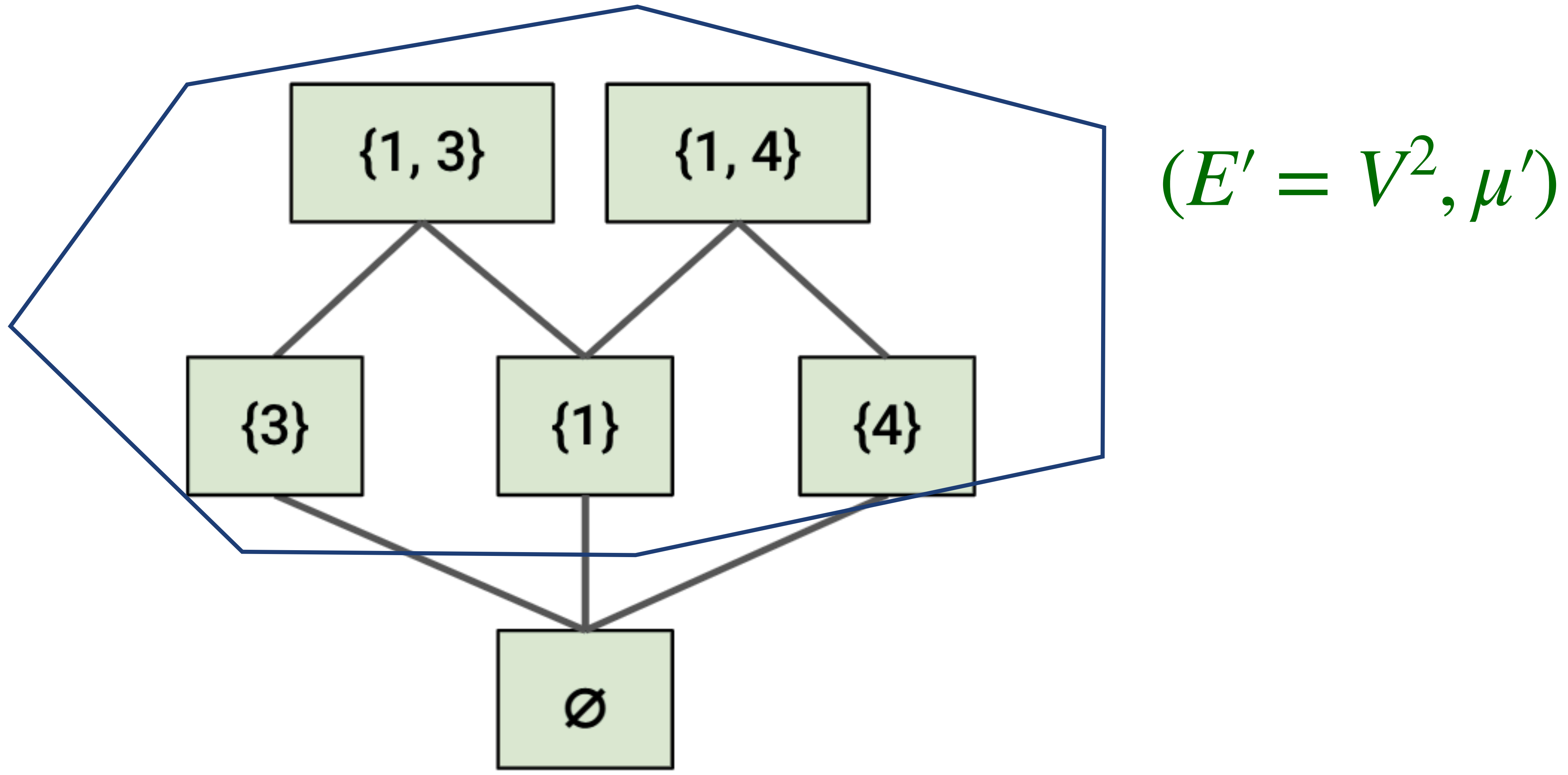

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$\forall$ link distribution $(V^{k-|S|}, \mu' = \mu_{s\to x})$

conditioned on coordinates in S being assigned x

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Example: ϵ high-dimensional expander

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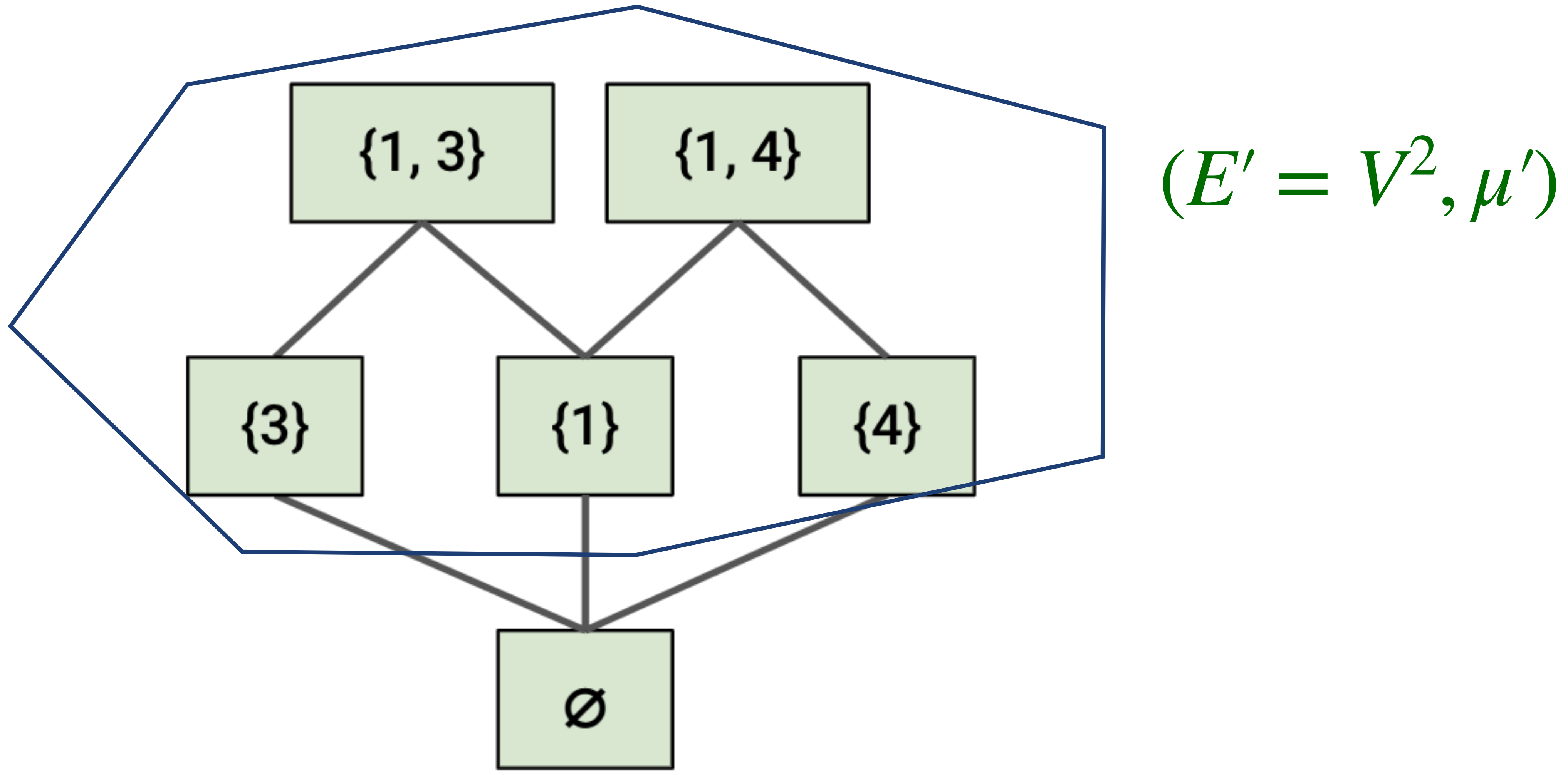
$\forall\ f, g : V \to \mathbb{R}$ and variables i, j in $[k]\backslash S$

$$\mathrm{Corr}(f,g) = \frac{\langle f - \mathbb{E}_{\mu_i'} f, g - \mathbb{E}_{\mu_j'} g \rangle_{\mu_{i,j}'}}{\|f - \mathbb{E}_{\mu_i'} f\|_{\mu_i'} \|g - \mathbb{E}_{\mu_j'} g\|_{\mu_j'}} \le \epsilon$$



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$$\lambda_2(\mathrm{Link}) = \max_{f,g} \frac{\langle f - \mathbb{E}_{\mu'^{(0)}} f, g - \mathbb{E}_{\mu'^{(0)}} g \rangle_{\mu'}}{\|f - \mathbb{E}_{\mu'^{(0)}} f\|_{\mu'^{(0)}} \|g - \mathbb{E}_{\mu'^{(0)}} g\|_{\mu'^{(0)}}} \le \epsilon$$

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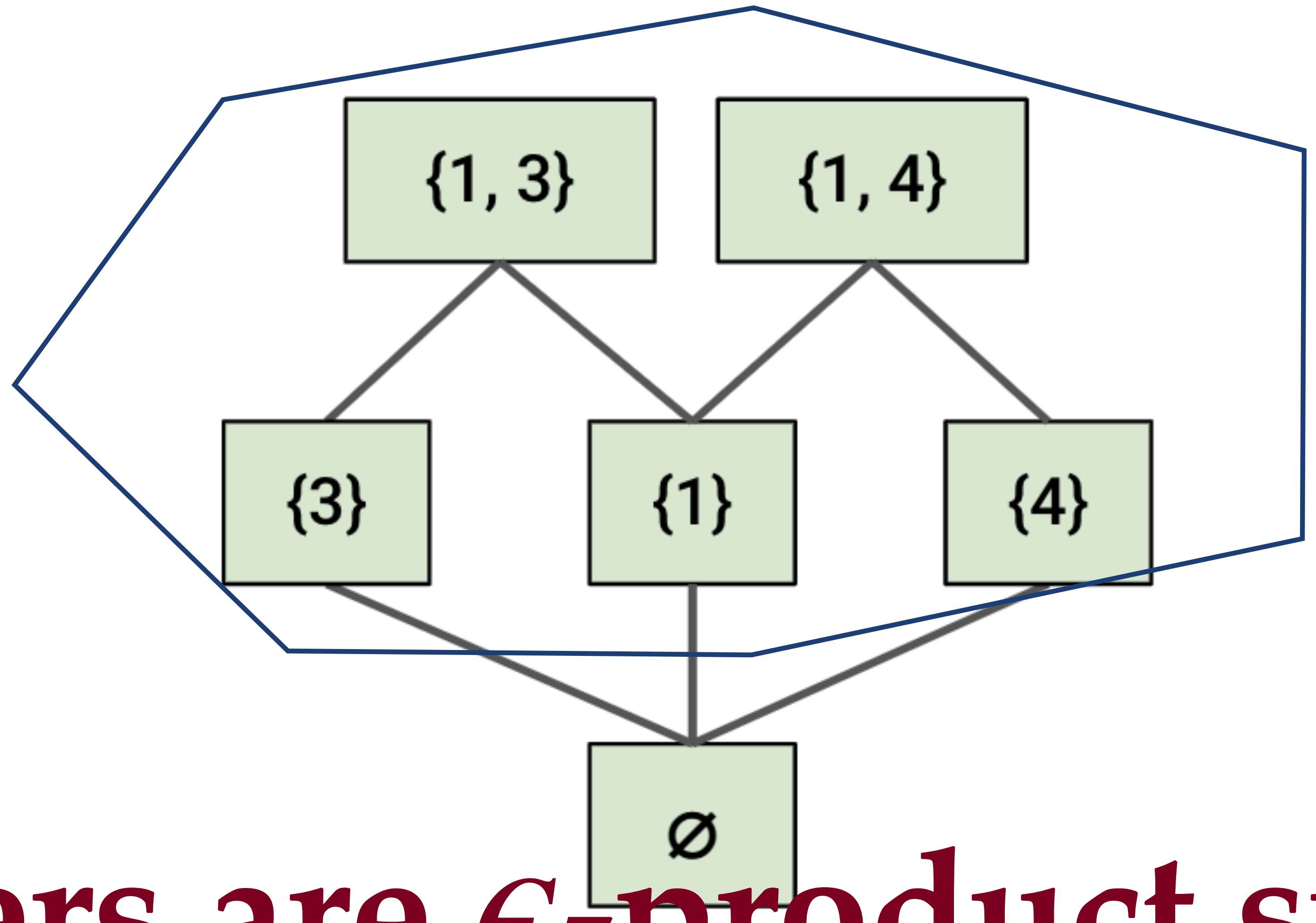

$(E' = V^2, \mu')$

ϵ high-dimensional expanders are ϵ-product spaces

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This talk

Theorem: For ϵ-product space (Ω, μ) and $f \in L^2(\mu)$
if f is deg-d and (d, δ)-global, then
$$\|f\|_4^4 \leq (400d)^d \delta \cdot \|f\|_2^2 + O_k(\epsilon\delta)\|f\|_2^2$$

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[BHKL]

Use walk operators defined on HDXs to obtain almost orthogonal decomposition

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This work

Use decomposition analogous to Efron-Stein decomposition over product spaces

Global functions

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General hypercontractivity $\not\Rightarrow$ **Weights of low density boolean functions concentrate on high degrees**

A function $f \in L^2(\mu)$ is (d, δ)-global if

$$\forall S \subseteq [k],\ |S| \leq d,\ \text{and } x \in V^S,$$

$$\|f\|_{\mu_{S \to x}}^2 \leq \delta$$

Useful notions for hypercontractivity

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An orthogonal decomposition of $f \in L^2(\mu)$

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Example: $\{\pm 1\}^n$, $f = \sum_{S \subseteq [k]} \widehat{f(S)}\, \chi_S$

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A derivative operator for $f \in L^2(\mu)$

$D_{S,x}f$ derivative wrt to variables in S, evaluated at $S \to x$

$D_{S,x}f$ has degree at most $\deg(f) - |S|$

Notions over product space

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Efron-Stein decomposition of $(V^k, (\mu^{(0)})^k)$

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$$f = \sum_{S \subseteq [k]} f^{=S}$$

$A_S f = \sum_{T \subseteq S} f^{=T} = \mathbb{E}_{\mu_{[k] \setminus S}} f$ depends only on coordinates in S

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$$D_{S,x} f(\cdot) = \sum_{T \supseteq S} f^{=T}(x, \cdot)$$

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+ induction on the deg of f

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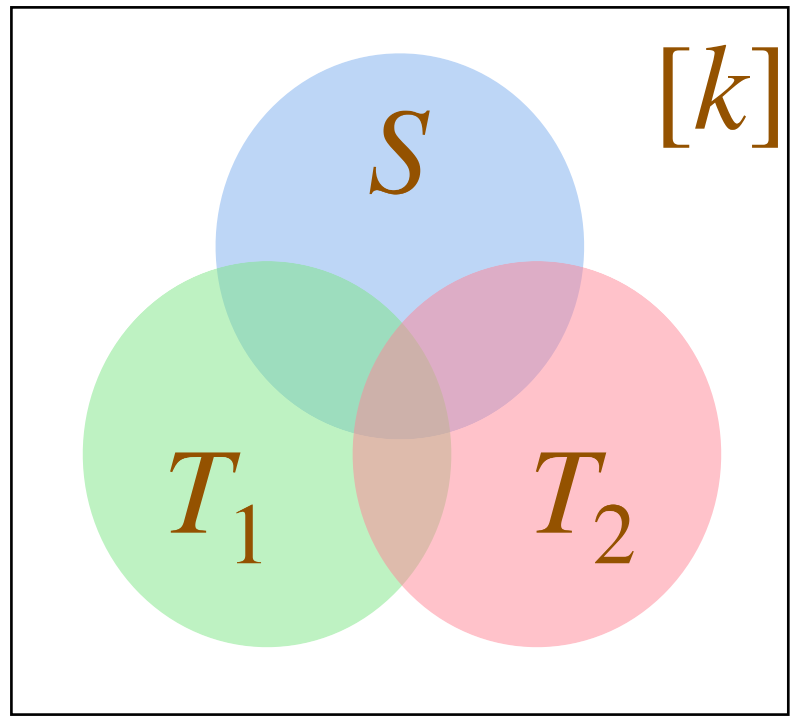

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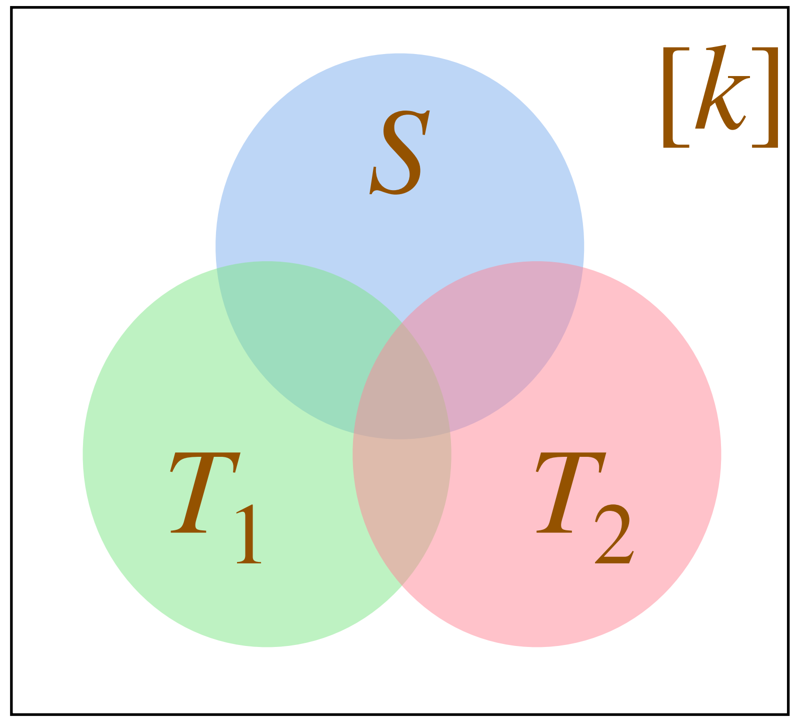

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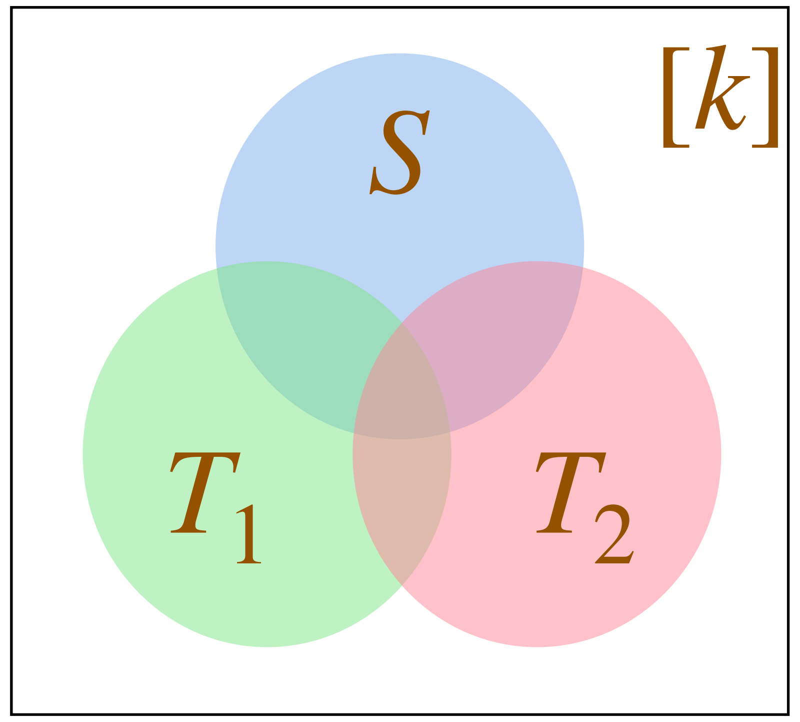

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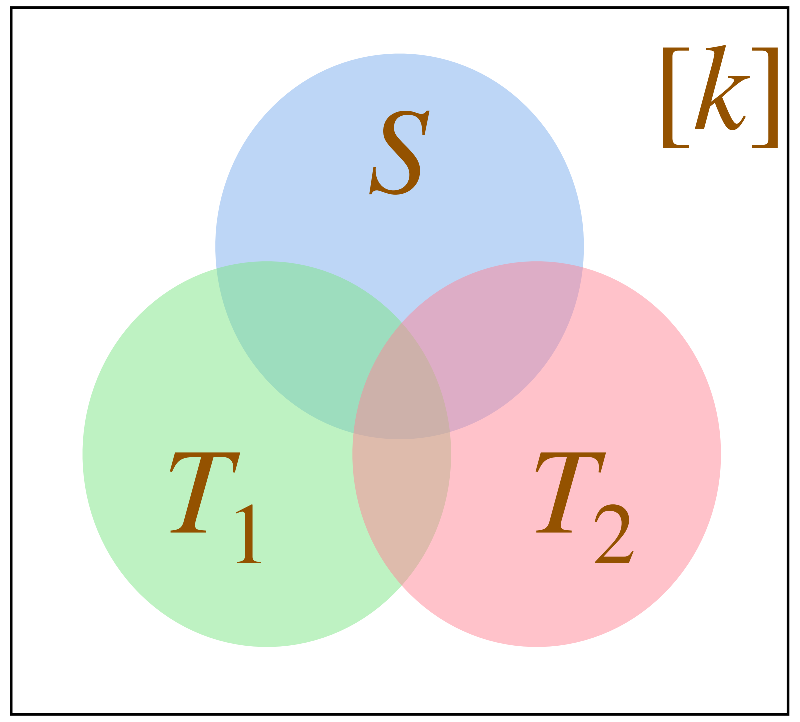

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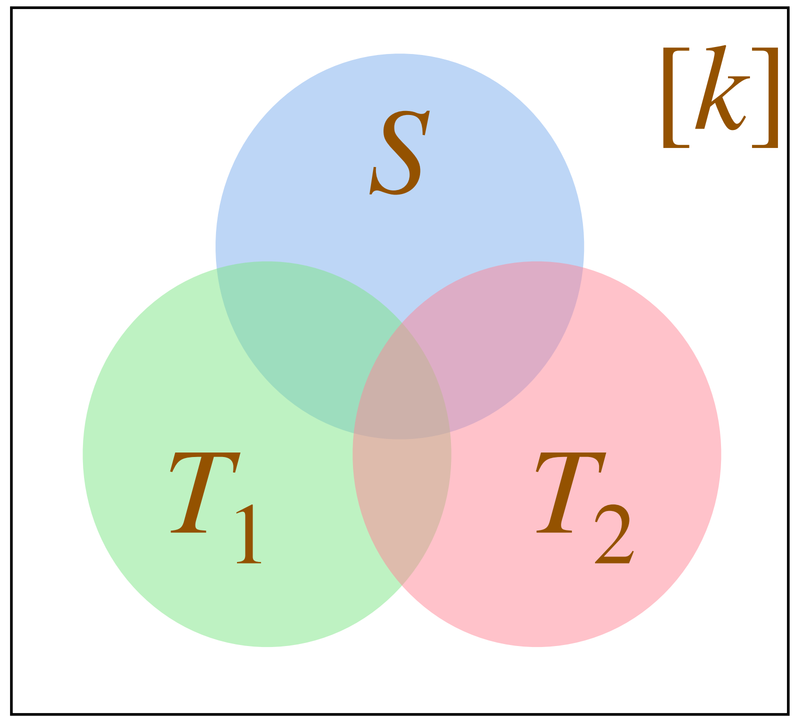

$T_1 \cap T_2 \cap S \neq \emptyset$ appears in $\mathbb{E}_{x\sim\mu_T}[\|D_{T,x}f\|_4^4]$ for $T \subseteq T_1 \cap T_2 \cap S$

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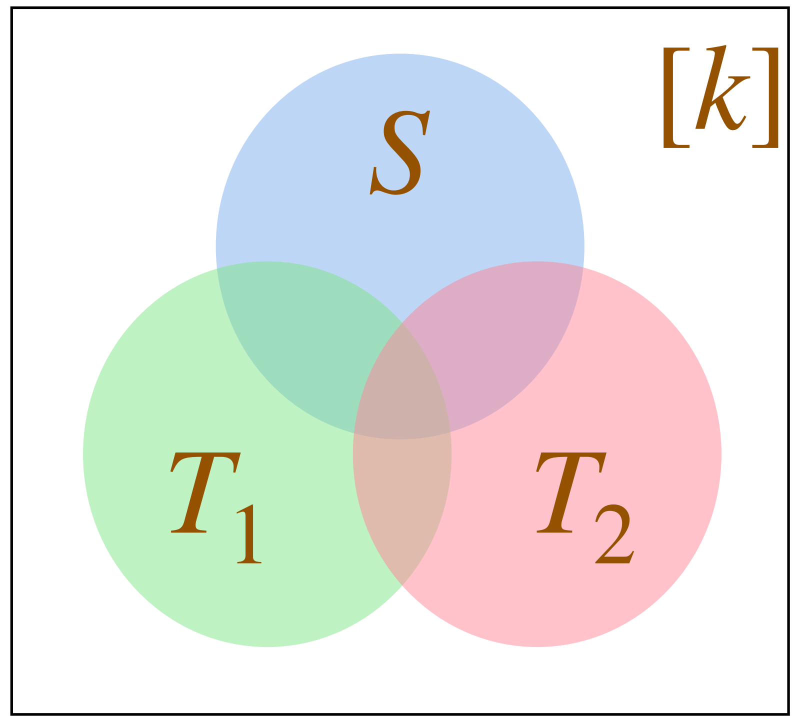

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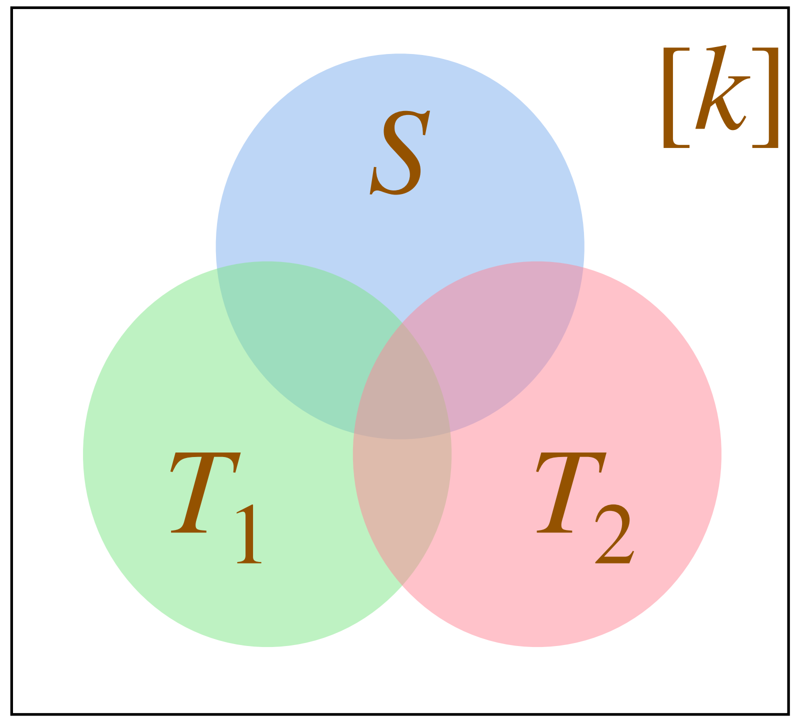

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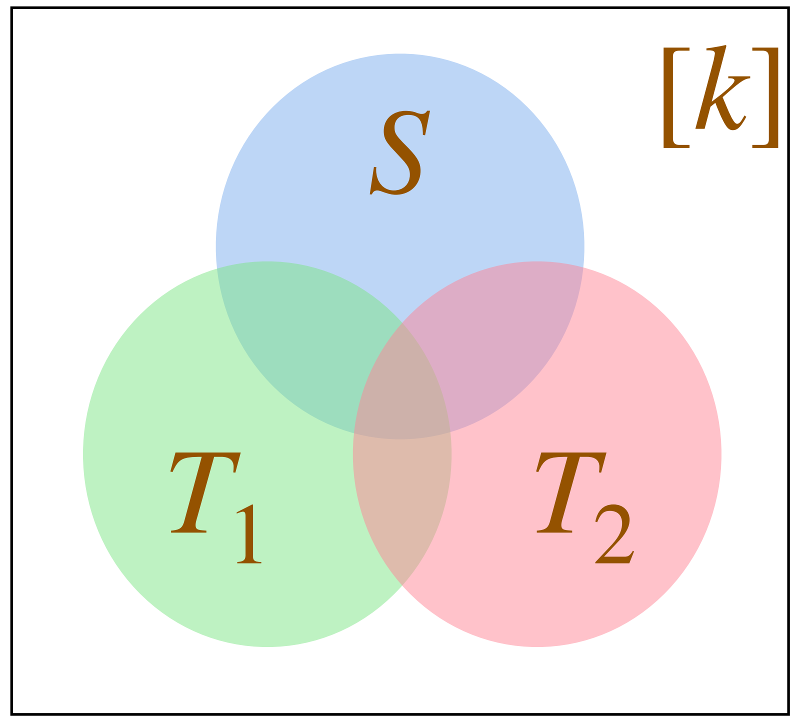

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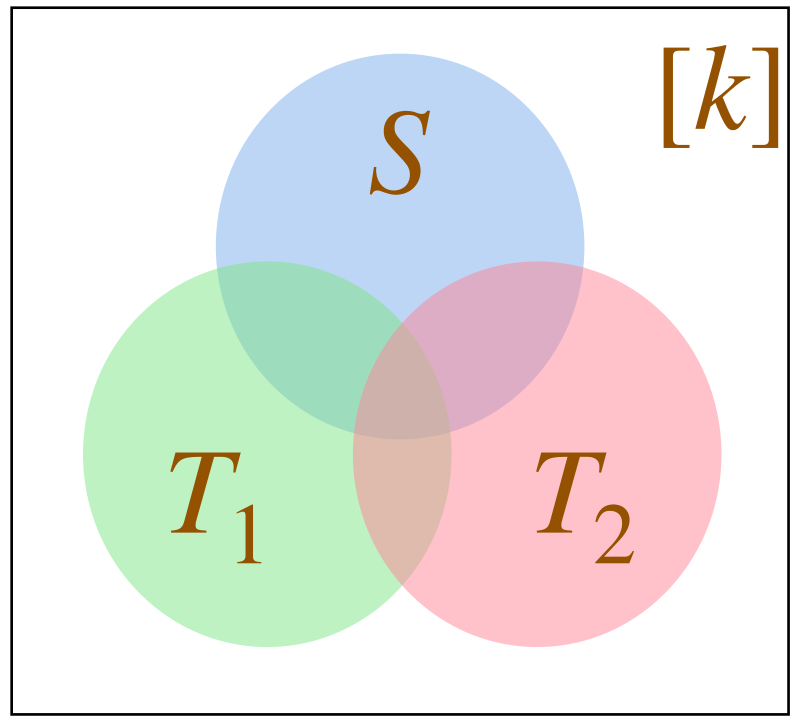

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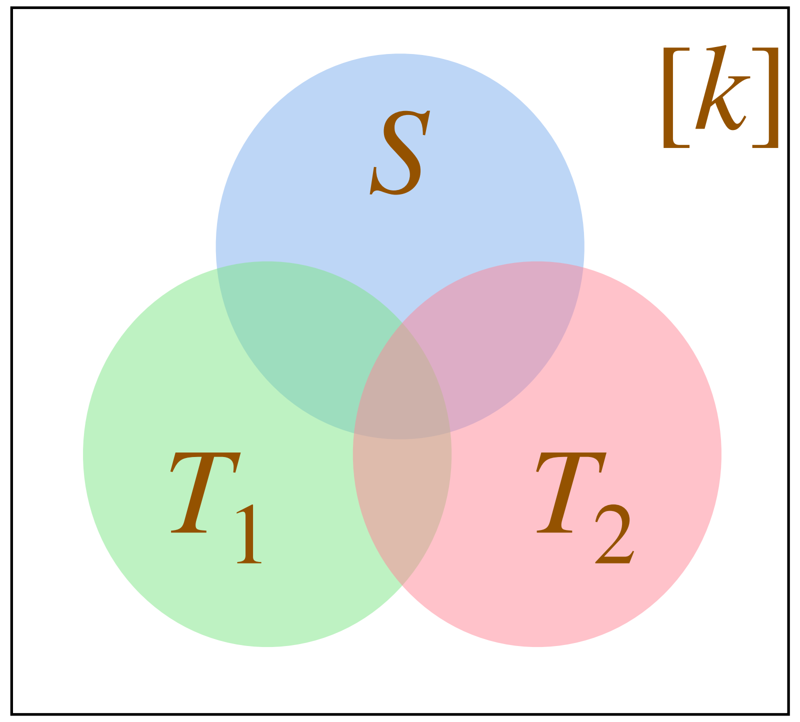

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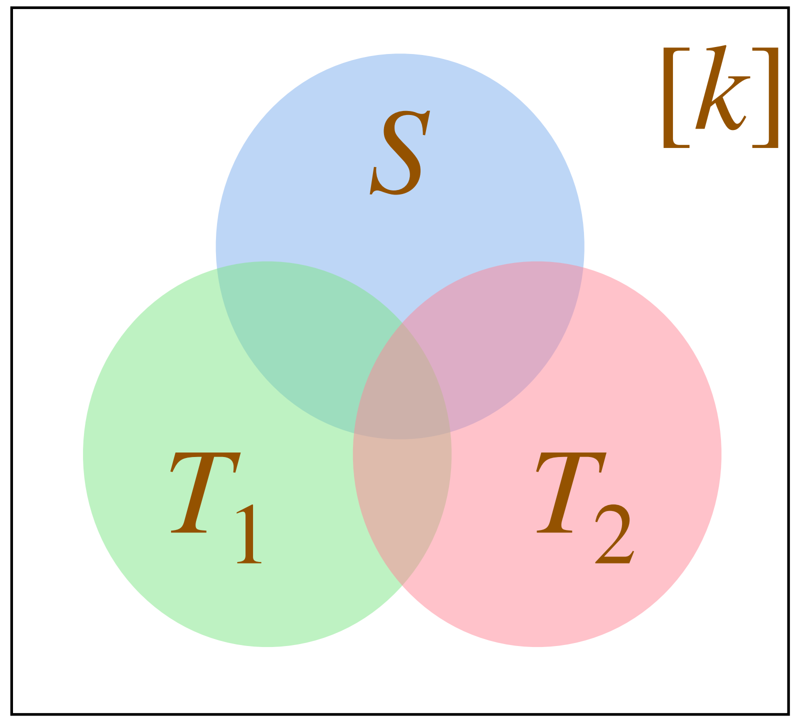

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$$(f^{=T_1}f^{=T_2})^{=S} = A_s(f^{=T_1}f^{=T_2}) = A_sf^{=T_1}A_sf^{=T_2}$$

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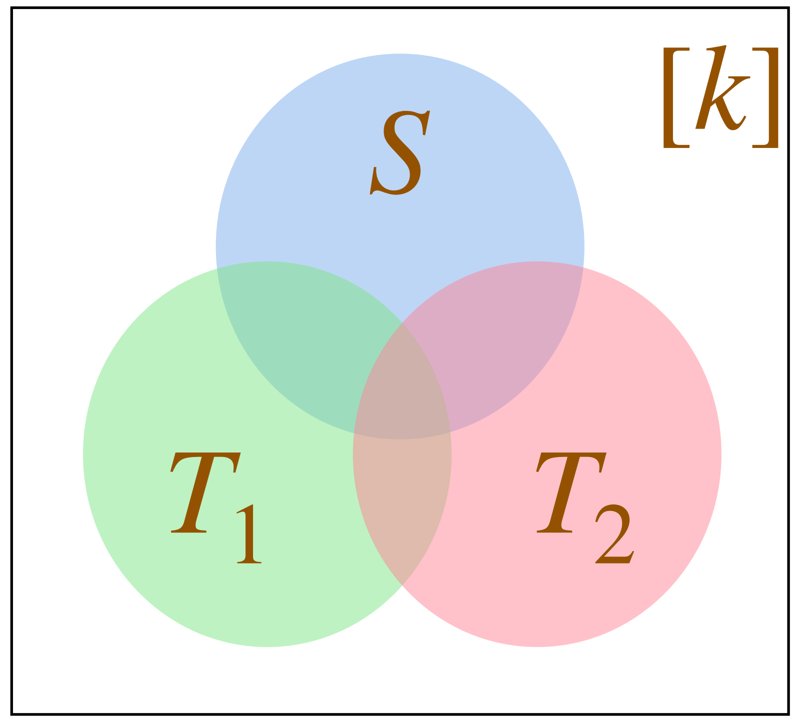

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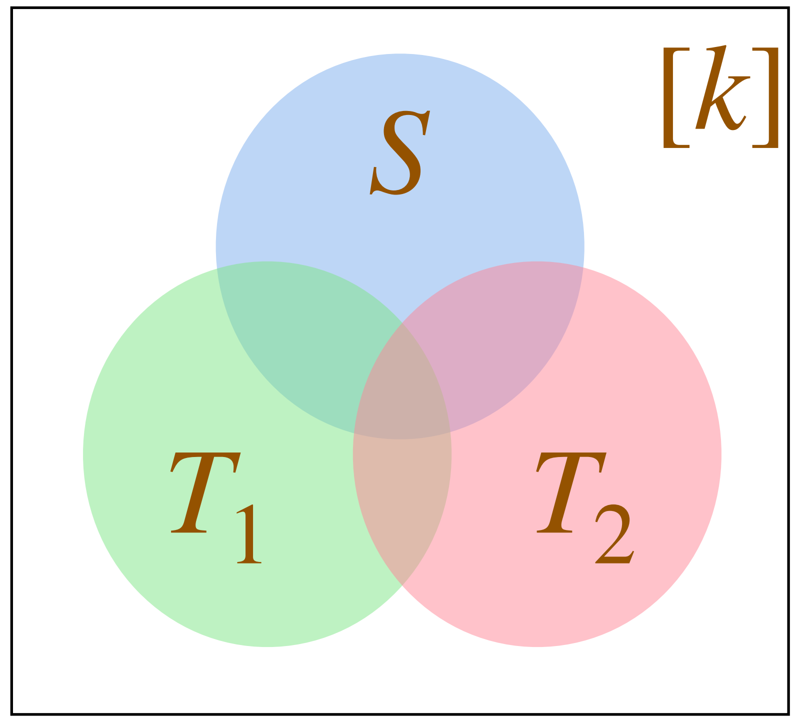
$\forall S' \subsetneq S$, $(f^{=T_1}f^{=T_2})^{=S'} = 0$ by case (2)

Key lemma:

$$\|f\|_4^4 \le 2\left(9^d\delta\|f\|_2^2 + \sum_{\varnothing\neq T\subseteq[k]} (4d)^{|T|}\mathbb{E}_{x\sim\mu_T}[\|D_{T,x}f\|_4^4]\right)$$

$$\|f\|_4^4 = \|f^2\|_2^2 = \sum_{S\subseteq[k]} \|(f^2)^{=S}\|_2^2 = \sum_{S\subseteq[k]} \left\| \sum_{T_1,T_2\subseteq[k]} \left(f^{=T_1}f^{=T_2}\right)^{=S} \right\|_2^2$$



$$\begin{cases} T_1 \cap T_2 \cap S \neq \varnothing \\ \exists i \in \text{exactly one of } T_1, T_2, S \\ T_1 \Delta T_2 = S \end{cases}$$

By unique decomp.

$$(f^{=T_1}f^{=T_2})^{=S} = A_s(f^{=T_1}f^{=T_2}) = A_sf^{=T_1}A_sf^{=T_2}$$

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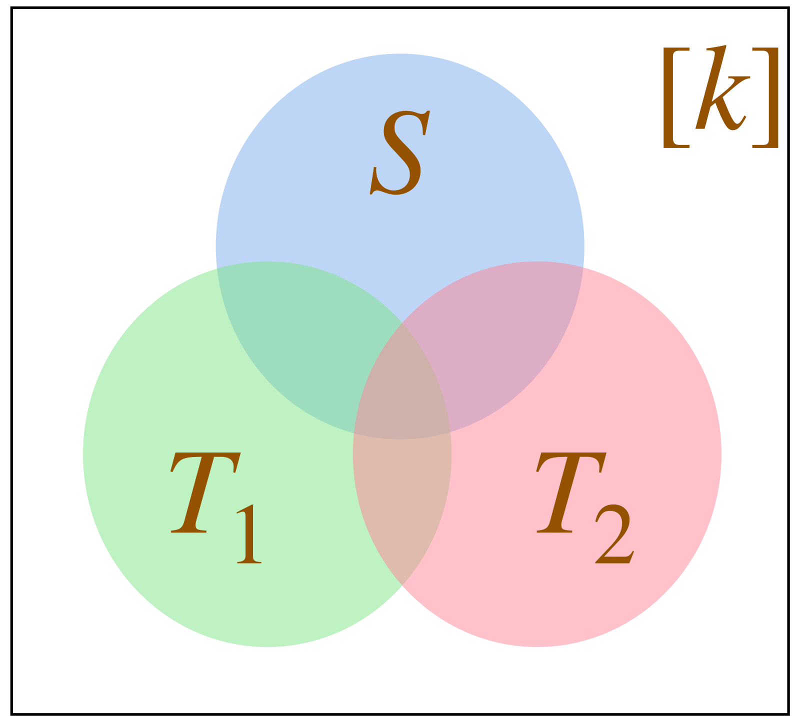

- $T_1 \cap T_2 \cap S \neq \emptyset$
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$A_s f^{=T_1}$ is a function over coordinates $S \cap T_1$

$A_s f^{=T_2}$ is a function over coordinates $S \cap T_2$

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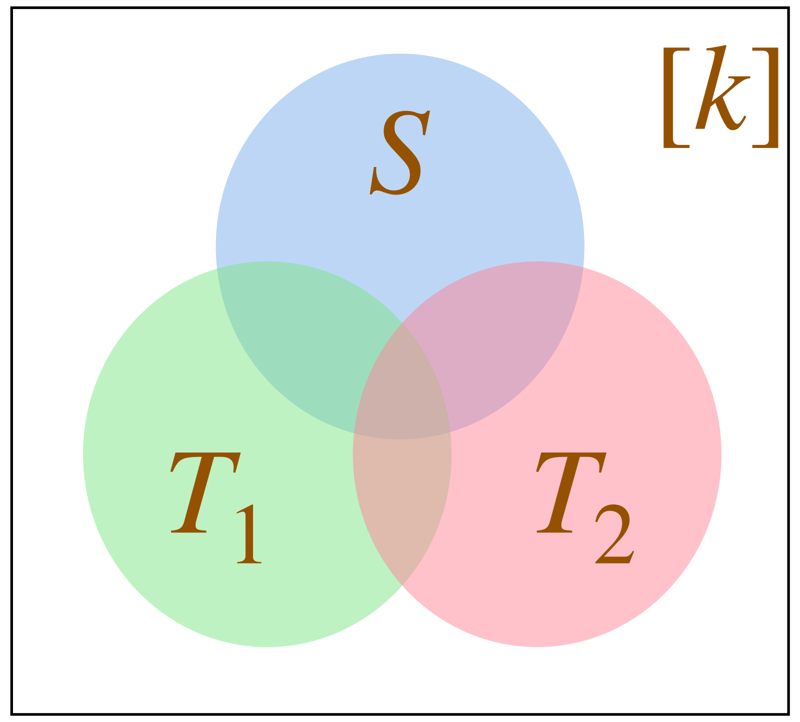

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$A_s f^{=T_1}$ is a function over coordinates $S \cap T_1$

$A_s f^{=T_2}$ is a function over coordinates $S \cap T_2$

$$\|(f^{=T_1} f^{=T_2})^{=S}\|_2 = \|A_s f^{=T_1}\|_2 \|A_s f^{=T_2}\|_2$$

$$\le \|f^{=T_1}\|_2 \|f^{=T_2}\|_2$$

Hypercontractivity over ϵ-product space

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Generalized Efron-Stein decomposition of $(V^k, (\mu^{(0)})^k)$

$$f = \sum_{S \subseteq [k]} f^{=S} \qquad A_S f = \mathbb{E}_{\mu_{[k]\setminus S}} f$$

ϵ-close to orthogonal

Different decompositions are close in $\|.\|_2$ distance

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$$D_{S,x} f(\cdot) = \sum_{T \subseteq S} (-1)^{|T|} A_{[k]\setminus T} f(x, \cdot)$$

$$\|D_{S,x} f - (D_{S,x} f)^{\leq \deg(f) - |S|}\|_2 \leq O_k(\epsilon) \|f\|_2$$

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The same proof goes through with error term $O_k(\epsilon\delta)\|f\|_2^2$!

Hypercontractivity over ϵ-product space

Theorem: For ϵ-product space (Ω, μ) and $f \in L^2(\mu)$
if f is deg-d and (d, δ)-global, then

$$\|f\|_4^4 \leq (400d)^d \delta \cdot \|f\|_2^2 + O_k(\epsilon\delta)\|f\|_2^2$$

$$\Uparrow$$

Key lemma:

$$\|f\|_4^4 \leq 2\left(9^d \delta \|f\|_2^2 + \sum_{\emptyset \neq T \subseteq [k]} (4d)^{|T|} \mathbb{E}_{x \sim \mu_T}[\|(D_{T,x} f)^{\leq d-|T|}\|_4^4]\right) + O_k(\epsilon\delta)\|f\|_2^2$$

+ induction on the deg of f

Open questions

Show (global) hypercontractivity for other spaces (coboundary expanders, other partially ordered sets, noncommutative probability space)

Improve the parameter C by considering T_ρ and/or stochastic processes