Working with Toni in Algebraic Proof Complexity
Something I like about proof complexity: gives a way of measuring the complexity of individual instances of SAT
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Unsaid: but actually, coming from computational/circuit complexity, I had a really hard time understanding and getting into proof complexity!
Why I Find Proof Complexity Too Hard

Too finicky about proofs:

What do you mean the Pigeonhole Principle and the Onto-Pigeonhole Principle aren’t just obviously equivalent?

Why should it matter whether I encode the pigeonhole principle using $\sum_j x_{ij} \geq 1$ or $\prod_j (x_{ij} - 1) = 0$? It’s the same principle!
Too syntactic:

“$\text{AC}^0$-Frege”? Where every line is an $\text{AC}^0$ formula? But as a function, every line is just “1”.

\[
\begin{align*}
\neg x \lor (x \land \neg y) \lor y & \quad \rightarrow \quad 1 \\
\neg y \lor y & \quad \rightarrow \quad 1 \\
1 & \quad \rightarrow \quad 1
\end{align*}
\]
2012-2014: I did a postdoc at U. Toronto.
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She tricked me! “Let’s just talk; you teach me something about algebraic circuits, I’ll teach you something about proof complexity, and we’ll see if we can come up with something to work on”
Bounded depth Frege = Frege where there’s a constant \( d \) s.t. proofs only ever uses the cut rule on formulas of depth \( d \).

Similarly for C-Frege for any syntactically-defined circuit class \( C \).

Okay, that made some sense to me!
Lines are of the form “f=0” (f a polynomial)

Various complexity measures:
- Max degree per line
- Total number of monomials
- Number of lines

Coming from algebraic circuit complexity: how to prove a lower bound on this? What polynomial even to prove bounds on (every proof has lots of lines)? It looks like a mess!
Input: An unsatisfiable system of polynomial equations
\[ F_1(\vec{x}) = F_2(\vec{x}) = \cdots = F_k(\vec{x}) = 0 \]

Hilbert’s Nullstellensatz: \( F_1 = F_2 = \cdots = F_k = 0 \) has no solutions if and only if there are polynomials \( G_1, \ldots, G_k \) such that
\[ F_1 G_1 + F_2 G_2 + \cdots + F_k G_k = 1. \]

Introduce new place-holder variables \( y_1, \ldots, y_k \), get a new polynomial
\[ C(y_1, \ldots, y_k, \vec{x}) = y_1 G_1(\vec{x}) + \cdots + y_k G_k(\vec{x}) \]
The Ideal Proof System \([P96, P98, GP14]\)

Definition \([GP14]\): \(C(\vec{y}, \vec{x})\) is an IPS certificate if

1. \(C(F(\vec{x}), \vec{x}) = 1\)
2. \(C(\vec{y}, \vec{x}) \in \langle y_1, ..., y_k \rangle\) (ideal in \(F[y_1, ..., y_k, x_1, ..., x_n]\))

Definition: The IPS complexity of an unsatisfiable system of equations is the optimum function complexity of any certificate.

E.g. algebraic circuit size, formula size, VNP, ...

Default: algebraic circuit size \((\text{no degree bound!})\)
Our First Work Together

July 2013: earliest email I could find with a draft of our Ideal Proof System paper
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Eric Allender:
(1) Suggests the name “Ideal Proof System” (thanks Eric!)
(2) Asks “If PIT is EF-provably easy, then does EF p-simulate IPS?” (Also Andy Drucker.) Turns out yes!
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Follow-up work on the Ideal Proof System

[Forbes-Shpilka-Tzameret-Wigderson ’16]: Lower bounds on C-IPS for small circuit classes C, by “powering up” algebraic circuit lower bounds

[Li-Tzameret-Wang ‘15]: Characterize ordinary Frege (up to quasipoly) by noncommutative formula IPS (follows our/Allender’s suggestion to show that PIT for this class is Frege-provable)

[Alekseev-Grigoriev-Hirsch-Tzameret ‘19]: “Cone proof system”, analogue of IPS for semi-algebraic proofs, connection w/ $\tau$ Conjecture

Additional works: [ST21], [AF21], [GHT22], [GP??]
[P96]: Introduced considering algebraic circuit size of the Nullstellensatz certificates. (“Hilbert-like IPS” or “IPS$_{\text{LIN}}$”, proved equivalent to IPS [FSTW16])

[P98]: Number of lines in PC, represent each line however* you want. (Proved equivalent to det-IPS [GP14].)
Toni’s questions [P96] eventually resolved:
1. Close the $\Theta(n)$ vs $\Omega(\sqrt{n})$ gap for PC degree for PHP. [R98]
2. Is $\Theta(\sqrt{n})$ the right bound for $PHP_n^m$ with $m$ large? No. [R98]
3. Nullstellensatz degree lower bound on random 3CNF? [BI99]
5. Tighten degree bound on simulation of Resolution by PC. ?
All about Cutting Planes

Proof Systems

- Cutting Planes stronger than Resolution
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{pcQ} \rightarrow \text{res} \)
- Cutting Planes stronger than Truth table
  - Source: \( cp \rightarrow \text{trecp} \rightarrow \text{treeres} \rightarrow \text{tp} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{treereslinF} \rightarrow \text{treeres} \rightarrow \text{tp} \)
- Cutting Planes stronger than Tree-like resolution
  - Source: \( cp \rightarrow \text{trecp} \rightarrow \text{treeres} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{treereslinF} \rightarrow \text{treeres} \)
- Cutting Planes stronger than Regular resolution
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \rightarrow \text{regres} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{pcQ} \rightarrow \text{res} \rightarrow \text{regres} \)
- Cutting Planes stronger than Ordered resolution
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \rightarrow \text{regres} \rightarrow \text{ordres} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \rightarrow \text{regres} \rightarrow \text{pearl} \rightarrow \text{ordres} \)
- Cutting Planes stronger than Pool resolution
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \rightarrow \text{poolres} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{pcQ} \rightarrow \text{res} \rightarrow \text{poolres} \)
- Cutting Planes stronger than Linear resolution
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \rightarrow \text{linres} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{pcQ} \rightarrow \text{res} \rightarrow \text{linres} \)
- Cutting Planes stronger than Tree-like Cutting Planes
  - Source: [subsystem]
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \rightarrow \text{regres} \rightarrow \text{ordres} \rightarrow \text{peb+ind} \rightarrow \text{trecp} \)
- Cutting Planes simulates Cutting Planes with Unary Coefficients
  - Source: [subsystem]
- Cutting Planes weaker than Semantic Cutting Planes
  - Source: [subsystem]
  - Source: \( \text{semanticscp} \rightarrow \text{cliquecolourineq} \rightarrow \text{cp} \)
- Cutting Planes stronger than Cutting Planes with Saturation
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{res} \rightarrow \text{saturationcp} \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{pcQ} \rightarrow \text{res} \rightarrow \text{saturationcp} \)
- Cutting Planes simulated by Stabbing Planes
  - Source: [citation needed]
- Cutting Planes simulates Stabbing Planes with Unary Coefficients
  - Source: [citation needed] On the Power and Dimensions of Branch and Cut
- Cutting Planes incomparable wrt Polynomial Calculus over \( \mathbb{F}_2 \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{pcF} \)
  - Source: \( \text{pcF} \rightarrow \text{nsF} \rightarrow \text{ts+ind} \rightarrow \text{cp} \)
- Cutting Planes incomparable wrt Nullstellensatz over \( \mathbb{F}_2 \)
  - Source: \( cp \rightarrow \text{unary}cp \rightarrow \text{php} \rightarrow \text{pcF} \rightarrow \text{nsF} \)
  - Source: \( \text{nsF} \rightarrow \text{ts+ind} \rightarrow \text{cp} \)
Toni’s questions eventually resolved:
1. Close the $O(n)$ vs $\Omega(\sqrt{n})$ gap for PC degree for PHP. [R98]
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3. Nullstellensatz degree lower bound on random 3CNF? [BI99]
5. Tighten degree bound on simulation of Resolution by PC. ?
Toni’s questions from P96 still open:

1. Does poly-degree IPS p-simulate Extended Frege? (Probably not. Prove it!)
2. Get PC to work well for SAT in practice (though, see Noriko Arai’s talk yesterday)
Back to Pitassi ’96/’98

Toni’s questions from P98 still open:

4. Ajtai/Krajicek representation-theoretic approach to uniform lower bounds deserves further study.

5. Conjecture: For a prime p, if IPS over GF(p) is p-bounded, then NP=coNP. (Can prove directly, avoiding PIT?)

6. Natural proofs-like barrier for proof complexity?
Joint w/ Toni, Nicola Galesi, Adrian She (to appear on arXiv momentarily)

Tensor Isomorphism:
• Verbose version a bottleneck to improving Graph Isomorphism
• Succinct version is GI-hard
• Many natural algebraic problems are TI-complete, e.g. Ring Isomorphism or local equivalence of quantum states
Algebraic Proof Complexity Of Tensor Isomorphism

How hard, really, could $\text{TI}$ be?

Are these tensors isomorphic?

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]

Psst: Proof complexity?

From my talk at Banff (2019)
Tricks
Returning the Favor

How hard, really, could TI be?

Are these tensors isomorphic?

\[
\begin{pmatrix}
1 & 1 \\
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1 & 0 \\
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\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]

Psst: Proof complexity?

Aimed at Toni

From my talk at Banff (2019)
Main Results [Galesi-G.-Pitassi-She ‘23]:
1. $\Omega(n)$ lower bound on PC degree for Tensor Iso
2. $O(1)$-degree PC proofs for non-isomorphism of bounded-rank tensors
3. PC can’t decide matrix rank, nor derive $AB=I$ from $BA=I$ in sub-linear degree
4. Conjecture: PC+$\text{Inv}$ can’t solve Tensor Iso either

Open:
Stronger lower bound? Note: no Boolean axioms here (obv. upper bound is $2^{O(n^2)}$).
Go back and look at Toni’s open questions from 1996/98!

Toni: still at the forefront of proof complexity
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What great things will Toni trick us into next?
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Happy Birthday Toni!