Certifying Combinatorial Solving Using Cutting Planes with Strengthening Rules

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The Success of Combinatorial Solving and Optimization

- Rich field of math and computer science
- Impact in other areas of science and also industry:
  - airline scheduling
  - logistics
  - hardware verification
  - donor-recipients matching for kidney transplants [MO12, BvdKM$^+$21]
- Typically very challenging problems (NP-complete or worse)
- Lots of effort last couple of decades spent on developing sophisticated so-called combinatorial solvers that often work surprisingly well in practice
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]

The Dirty Little Secret...

- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22]
- Even worse: No way of knowing for sure when errors happen
- How to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)
- And solvers even get feasibility of solutions wrong (though this should be straightforward!)

What Can Be Done About Solver Bugs?

- **Software testing**
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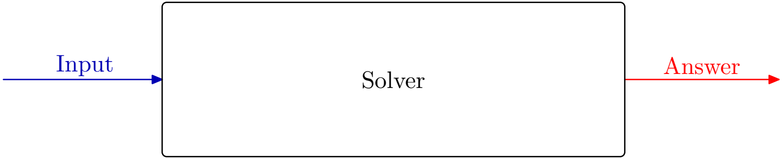
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by outputting
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run solver on problem input

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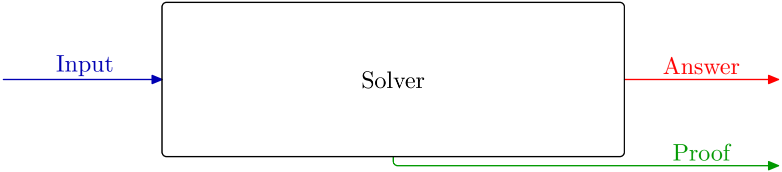

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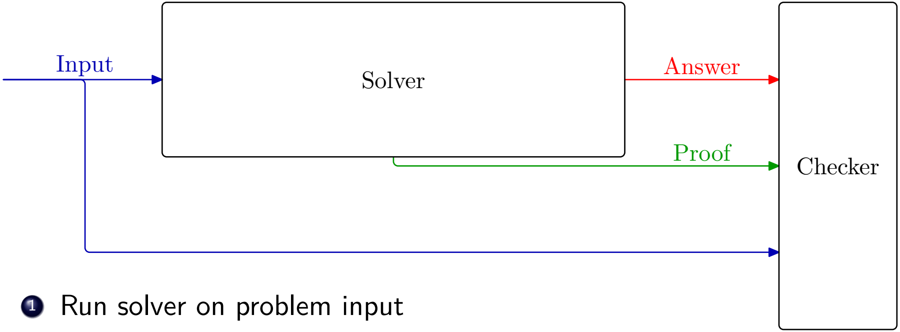

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3. Feed input + answer + proof to proof checker

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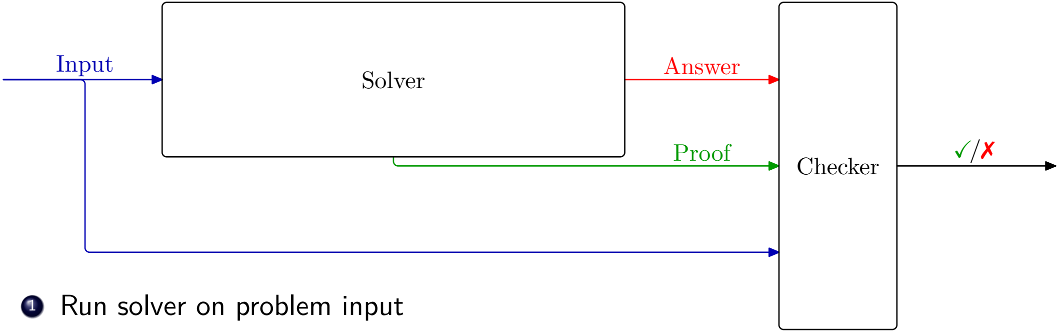

1. Run solver on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging with Certifying Solvers: Requirements

Proofs produced by certifying solver [ABM$^+$11, MMNS11] should

- be powerful enough to allow proof logging with minimal overhead
- be simple enough to make proof checking very easy
- not require knowledge of inner workings of solver

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Con: Clear conflict expressivity vs. simplicity!

Pro: Does not prove solver correct, but proves solution correct
Proof checker can be simple enough to be formally verified

The Sales Pitch for Proof Logging

1. ***Certifies correctness*** of solver output
2. ***Detects errors*** even if due to compiler bugs, hardware failures, or cosmic rays
3. Helps with ***debugging*** during development [EG21, GMM$^+$20, KM21, BBN$^+$23]
4. Facilitates ***performance analysis***
5. Helps identify potential for ***further improvements***
6. Enables ***auditability*** by third parties
7. Serves as stepping stone towards ***explainability***

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Success story for SAT solving: DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH⁺17], ...

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Success story for SAT solving: DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH$^+$17], ...

But has remained out of reach for stronger paradigms
And, in fact, even for some advanced SAT solving techniques

This Talk

If we use

- 0–1 linear inequalities instead of clauses
- cutting planes instead of resolution
- well-chosen strengthening rules

we get general-purpose proof system for combinatorial optimization!

Outline of This Talk

1. Basic SAT Solving
   - CDCL by Example
   - Resolution
   - Extension Rules
2. Advanced SAT Techniques
   - Cardinality Constraints and Pseudo-Boolean Reasoning
   - Translating Pseudo-Boolean Constraints to CNF
   - Parity Reasoning
3. Beyond SAT
   - Constraint Programming
   - Symmetry, Dominance, and Optimization
   - Formal Proof System

The SAT Problem

- Variable x: takes value **true** $(=1)$ or **false** $(=0)$
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge$$
$$(x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

A Quick Recap of Modern SAT Solving

DPLL method [DP60, DLL62]

- Assign values to variables (in some smart way)
- Backtrack when conflict with falsified clause

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Conflict-driven clause learning (CDCL) [MS96, BS97, MMZ+01]

- Analyse conflicts in more detail — add new clauses to formula
- More efficient backtracking
- Also let conflicts guide other heuristics

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Variable Assignments

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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Add to assignment trail

Until satisfying assignment or conflict

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$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \vee r}{=} 1$
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} decision level 2

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$z \stackrel{x \vee \overline{y} \vee z}{=} 1$
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} decision level 3

Decision

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Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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decision level 3

Could backtrack by removing last decision level & flipping last decision

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$\left.\begin{array}{l} \boxed{p \stackrel{\mathsf{d}}{=} 0} \\ u \stackrel{p \lor \overline{u}}{=} 0 \end{array}\right\}$ decision level 1

$\left.\begin{array}{l} \boxed{q \stackrel{\mathsf{d}}{=} 0} \\ r \stackrel{q \lor r}{=} 1 \\ w \stackrel{\overline{r} \lor w}{=} 1 \end{array}\right\}$ decision level 2

$\left.\begin{array}{l} \boxed{x \stackrel{\mathsf{d}}{=} 0} \\ y \stackrel{u \lor x \lor y}{=} 1 \\ z \stackrel{x \lor \overline{y} \lor z}{=} 1 \\ \stackrel{\overline{y} \lor \overline{z}}{\bot} \end{array}\right\}$ decision level 3

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But want to learn from conflict and cut away as much of search space as possible

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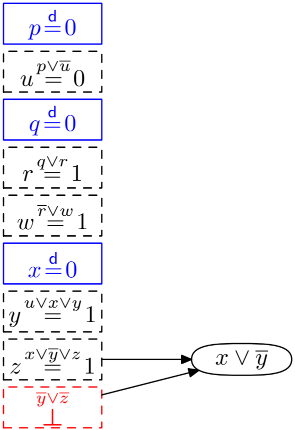

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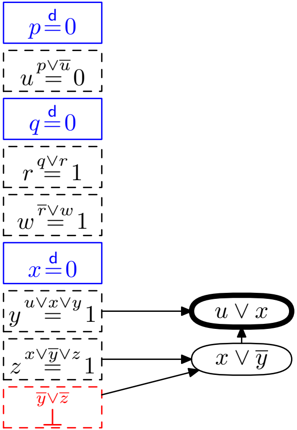
Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Could backtrack by removing last decision level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
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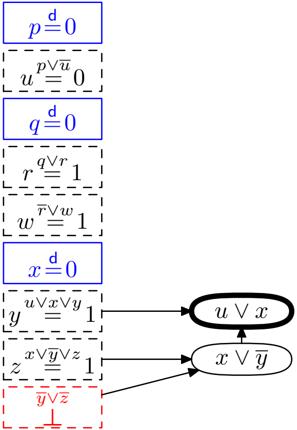
Repeat until UIP clause with only 1 variable after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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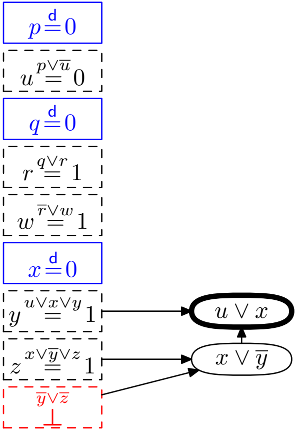

Assertion level 1 (max for non-UIP literal in learned clause) — trim trail to that level

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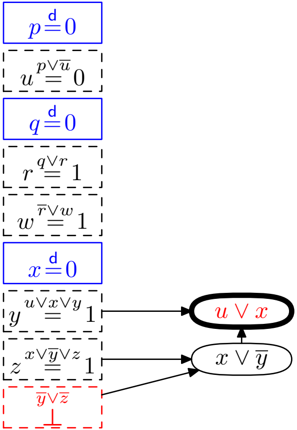

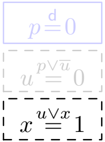

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Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

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Backjump: undo max #decisions while learned clause propagates

$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

$u \lor x$

$x \lor \overline{y}$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$x \stackrel{u \lor x}{=} 1$

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Then continue as before...

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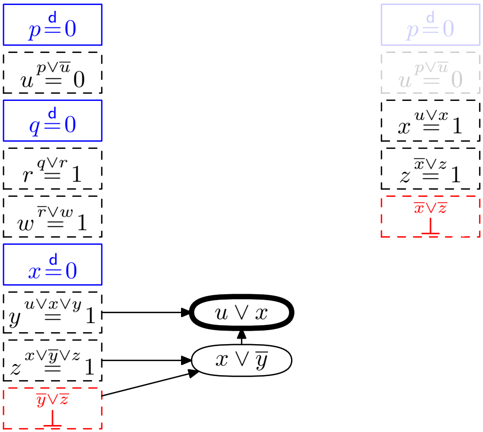

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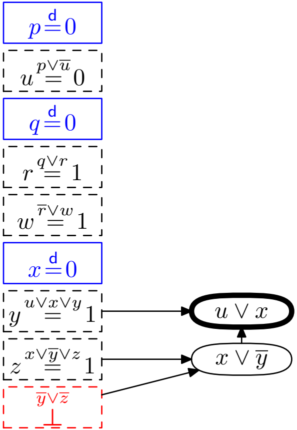

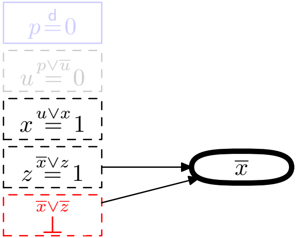

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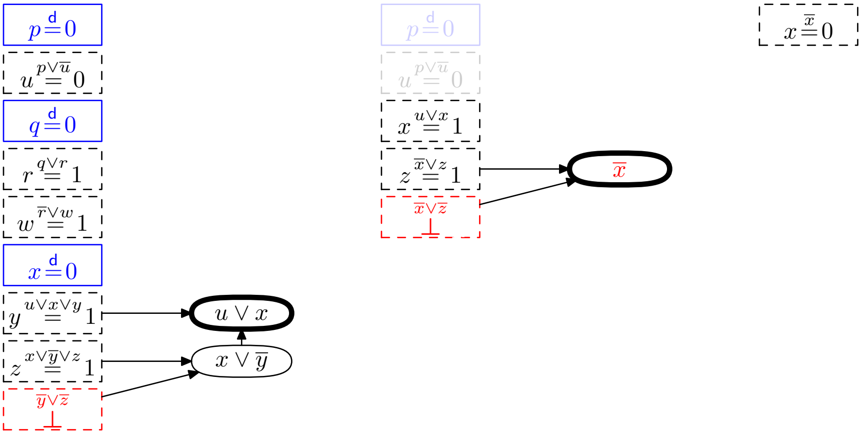

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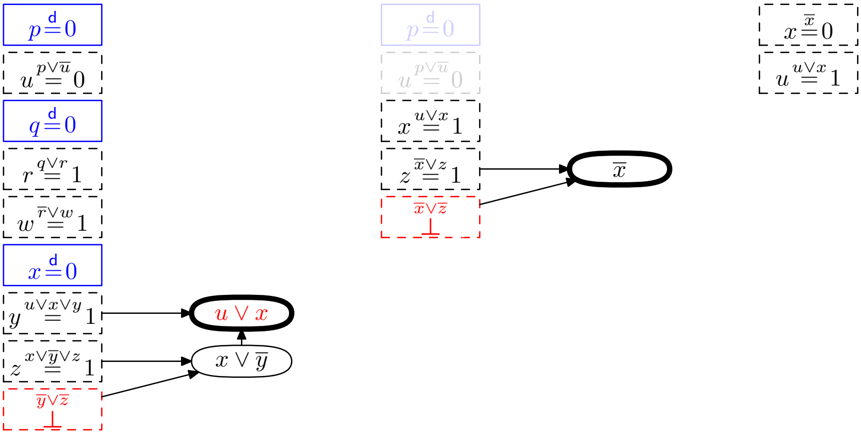

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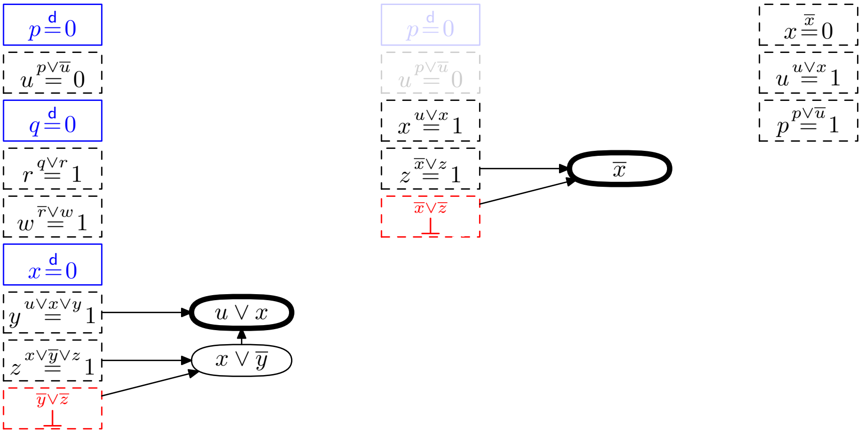

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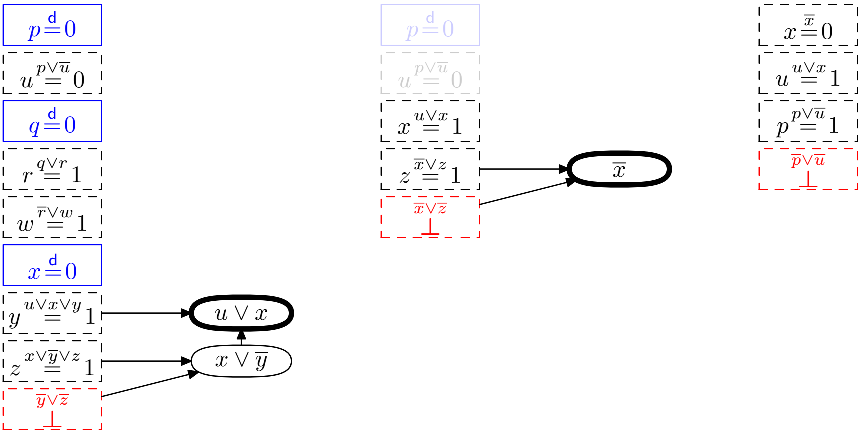

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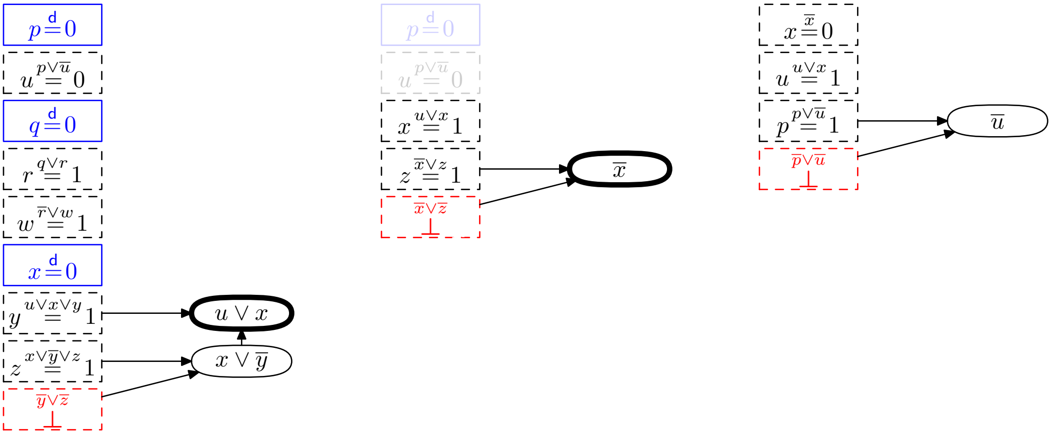

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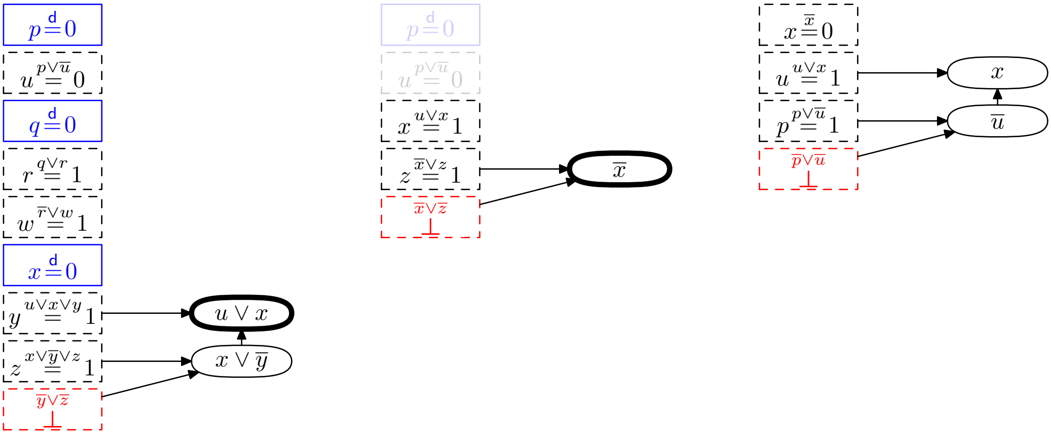

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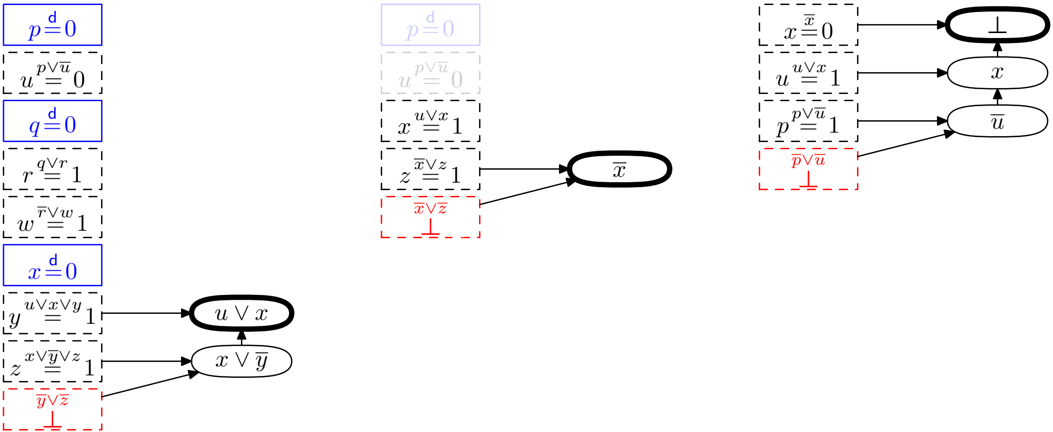

CDCL Reasoning and the Resolution Proof System

For CDCL proof logging, need proof system for unsatisfiable formulas

Focus on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

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- Derive new clauses by **resolution rule**

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(*) Ignores pre- and inprocessing, but we will get there...

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Obtain resolution proof. . .

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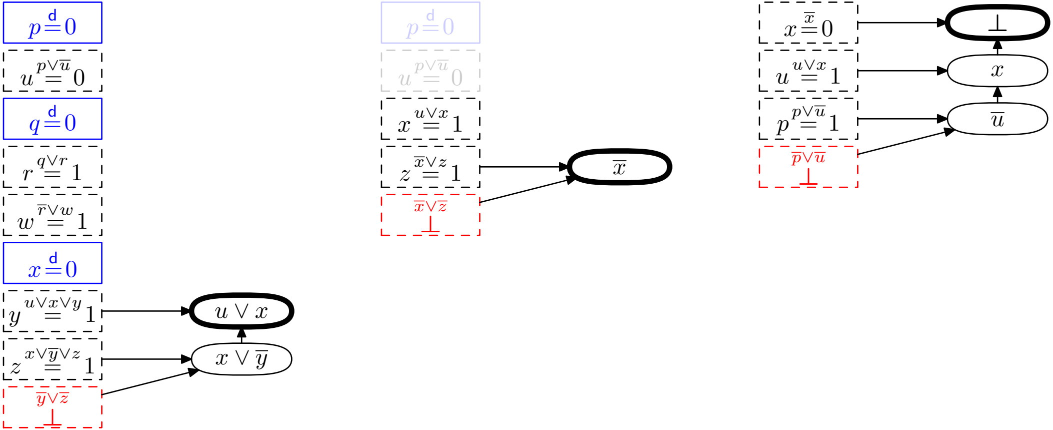
Obtain resolution proof from our example CDCL execution...



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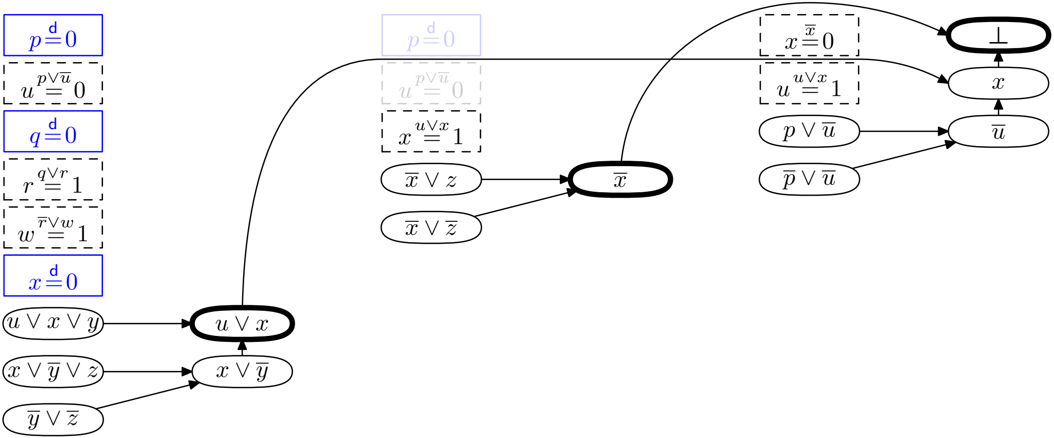
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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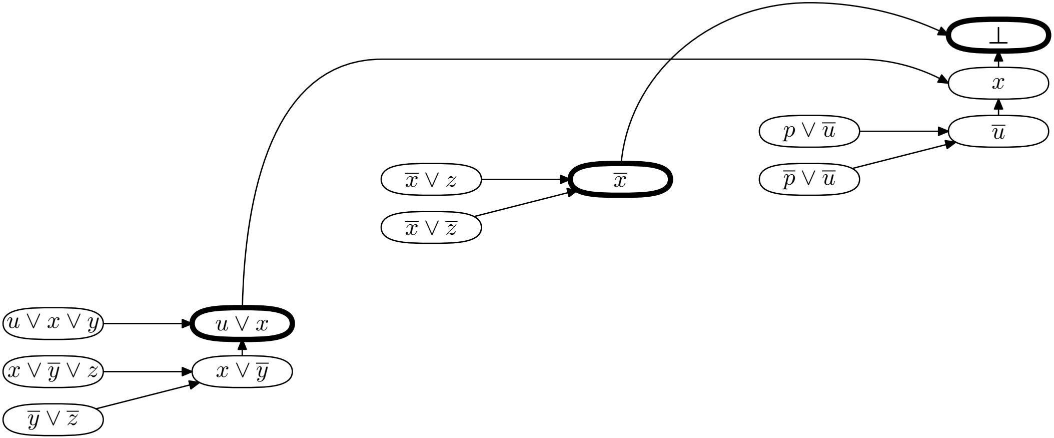

Extension Variables and Redundant Clauses

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CDCL pre- and inprocessing techniques can do steps like this

But resolution proof system cannot certify such derivations (by definition)

Redundance-Based Strengthening

- C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable
- Adding redundant clauses should be OK
- Previous rules such as RAT [JHB12] and propagation redundancy [HKB17]

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$
- Implication should be efficiently verifiable (ω specified; derivations of clauses in $(F \wedge C){\restriction_\omega}$ explicitly given or truly obvious)

Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

using redundance-based strengthening condition $F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$

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1. $F \wedge \neg(a \vee \overline{x} \vee \overline{y}) \models (F \wedge (a \vee \overline{x} \vee \overline{y}))\restriction_{\omega}$

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3. $F \wedge (a \vee \overline{x} \vee \overline{y}) \wedge (\overline{a} \vee x) \wedge \neg(\overline{a} \vee y) \models (F \wedge (a \vee \overline{x} \vee \overline{y}) \wedge (\overline{a} \vee x) \wedge (\overline{a} \vee y))\restriction_\omega$

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   Choose $\omega = \{a \mapsto 0\}$ — F untouched; new clause satisfied
   $\omega = \{a \mapsto 0\}$ also satisfies $\overline{a} \vee x$
   $\neg(\overline{a} \vee y)$ forces $y \mapsto 0$ which satisfies $a \vee \overline{x} \vee \overline{y}$

Proof Logging for State-of-the-Art SAT Solving

Resolution + redundance rule is as strong as extended Frege proof system

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Except

- really care about efficiency
- for some advanced techniques don't know efficient proof logging methods

Reasoning with Cardinality Constraints

Given clauses

$$x_1 \lor x_2 \lor x_3$$
$$x_1 \lor x_2 \lor x_4$$
$$x_1 \lor x_3 \lor x_4$$
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can deduce that

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can deduce that

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How provide proof logging for reasoning with such cardinality constraints?

Can solve pigeonhole principle efficiently — exponentially hard for CDCL [Hak85, BKS04]

Implemented in LINGELING [Lin], but not with DRAT proof logging
Resolution + extension rule can do it in theory, but efficiently in practice?!

Pseudo-Boolean Constraints

Pseudo-Boolean constraints are 0-1 integer linear inequalities

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- as before, variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

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$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

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Lin comb $$\frac{w + 2x + 4y + 2z \geq 5 \qquad w + 2x + y \geq 2}{}$$

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Recovering cardinality constraints from CNF

Clauses

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Pseudo-Boolean constraints

$$x_1 + x_2 + x_3 \geq 1$$
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$$x_1 + x_3 + x_4 \geq 1$$
$$x_2 + x_3 + x_4 \geq 1$$

Add all up

$$3x_1 + 3x_2 + 3x_3 + 3x_4 \geq 4$$

and divide by 3 to get

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

Extended Cutting Planes

Combine cutting planes method with **redundance** rule

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Add constraint C to formula F if exists witness substitution ω such that

$$F \land \neg C \models (F \land C)\restriction_{\omega}$$

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- Cutting planes can do efficiently anything that resolution can do
- Reverse unit propagation works also for 0-1 linear inequalities
- RAT $=$ redundance rule with witness flipping RAT literal

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- Lifts reasoning from clauses to 0-1 inequalities
- Implemented in proof checker VeriPB [Ver, GN21, BGMN22]
- Yields surprisingly expressive proof logging system

CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- MiniSat+ [ES06]
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How know translation correct?
VeriPB can certify pseudo-Boolean-to-CNF rewriting [GMNO22]

Parity (XOR) Reasoning

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$
$$y \lor \overline{z} \lor \overline{w}$$
$$\overline{y} \lor z \lor \overline{w}$$
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want to derive

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But used in CryptoMiniSat [Cry]

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Add XORs to proof language?
Prefer to keep things super-simple and verifiable. . .

Pseudo-Boolean Proof Logging for XOR Reasoning

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Use redundance rule with fresh variables a,b to derive

$$x + y + z + 2a = 3$$
$$y + z + w + 2b = 3$$

("$=$" syntactic sugar for "$\geq$" plus "$\leq$")

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VeriPB can certify XOR reasoning [GN21]

Proof Logging for Graph Solving and Constraint Programming

Pseudo-Boolean proof logging can also certify reasoning in

- graph solvers without knowing what a graph is [GMN20, GMM^{+}20]
- constraint programming solvers without knowing what an integer is [EGMN20, GMN22]

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- constraint programming solvers without knowing what an integer is [EGMN20, GMN22]

Caveat: Input pre-translated into 0–1 integer linear program
This translation should be formally verified (work in progress)

Integer Variables

Represent integer a as sum of bits $\sum_i 2^i \cdot a_i$

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Use redundance-based strengthening to introduce new variables

$$a_{\geq k} \Leftrightarrow \sum_i 2^i \cdot a_i \geq k$$

$$a_{=k} \Leftrightarrow (a_{\geq k} \wedge \overline{a}_{\geq k+1})$$

(definitions representable as 0–1 inequalities)

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(definitions representable as 0–1 inequalities)

Go back and forth between representations to support efficient proof logging

Constraint Programming Reasoning

Efficient proof logging support for

- All-different propagators
- Table constraints
- Arrays
- Problem reformulations
- Backtracking during search
- Et cetera...

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Efficient proof logging support for

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- Et cetera. . .

Not at all trivial to implement
Lots of work left to get to full-fledged constraint programming solver
But so far everything has been possible to do [EGMN20, GMN22]

The Challenge of Symmetries

Symmetries crucial for some optimization problems [AW13, GSVW14]
Show up also in hard SAT benchmarks

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Symmetry breaking

- Add clauses filtering out symmetric solutions [DBBD16]
- DRAT proof logging for limited cases only [HHW15]

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Symmetric learning

- Allow to add all symmetric versions of learned clause [DBB17]
- Recently proposed proof logging in [TD20]
  1. Special-purpose, specific approach
  2. Requires adding explicit concept of symmetries
  3. Not compatible with preprocessing techniques

Better to keep proof system super-simple and verifiable. . .

Optimization Problems

Deal with symmetries by switching focus to optimization

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Note that $\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)$ means $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$

Spoiler alert:
For decision problem, nothing stops us from inventing objective function (like lexicographic order $\sum_{i=1}^{n} 2^i \cdot x_i$)

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Redundance-based strengthening, optimization version [BGMN22]

Add constraint C to formula F if exists witness substitution ω such that

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Can be more aggressive if witness ω strictly improves solution

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7. ...
8. Can't go on forever, so finally reach α' satisfying $F \land D$

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified) [BGMN22]

If $D_1, D_2, \ldots, D_{m-1}$ have been derived from F (maybe using dominance), then can derive also D_m if exists witness substitution ω such that

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- Switch between different orders in same proof

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Slightly simplified version of proof system — see [BGMN22] for full details

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(Ignore rules for improving solutions here — focus on decision problems)

Strengthening Rules

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- Witness ω should be explicitly specified
- For all right-hand side proof targets derivations should be specified or be truly obvious (e.g., by weakening)

Deletion, Core Transfer, and Order Change

Deletion

- No restrictions on deletions from derived set $\mathcal{D}$
- Delete C from $\mathcal{C}$ only if C can be derived from $\mathcal{C} \setminus \{C\}$ by
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- Except possible to add special cases for decision problems — see [BGMN22]

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Core transfer

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Change of order

Possible to change order if $\mathcal{D} = \emptyset$

Directions for Future Research

Proof logging for combinatorial optimization

- Pseudo-Boolean optimization and MaxSAT solving (work in [VDB22, BBN⁺23])
- General constraint programming
- Mixed integer linear programming (work in [CGS17, EG21])
- Satisfiability modulo theories (SMT) solving

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- We're hiring! Talk to me to join the proof logging revolution!

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like a promising approach
- Requires powerful but simple proof systems — need for "constructive proof complexity"
- Cutting planes with strengthening rules seems to hit a sweet spot
- Raises new and interesting questions also in "standard proof complexity"

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Thank you for your attention!

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