Rado numbers: SAT methods and connections to Nullstellensatz complexity

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Outline of Talk

1. Arithmetic Ramsey theory
2. Rado numbers and SAT
3. Nullstellensatz certificates for Ramsey-type numbers

Introduction

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Origins in algebra and number theory

Origins of Ramsey Theory

Theorem (Hilbert's cube lemma, 1892)

For every k and d, there is an n such that every k-coloring of $\{1, \ldots, n\}$ produces a monochromatic solution to the system

$$x_0 + \sum_{i \in I} x_i = x_I, \quad I \subseteq \{1, \ldots, d\}, I \neq \emptyset.$$

Hilbert used this to prove results on irreducibility of rational functions.

Schur's Theorem

Theorem (Schur, 1916)

For every $k \geq 1$, there exists a number n such that every k-coloring of $\{1, 2, \ldots, n\}$ contains a monochromatic triple (x, y, z) satisfying

$$x + y = z.$$

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- The Schur number $S(k)$ is the *smallest* such n

Schur numbers

- $S(2) = 5$

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- We can 2-color [4] while avoiding monochromatic solutions to $x + y = z$:

1 2 3 4

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- $S(2) = 5$
- We can 2-color [4] while avoiding monochromatic solutions to $x + y = z$:

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- $S(3) = 14$:
- We can 3-color [13] while avoiding monochromatic solutions to $x + y = z$:

1 2 3 4 5 6 7 8 9 10 11 12 13

Van der Waerden's Theorem

Theorem (van der Waerden, 1927)

For every $k, \ell \geq 1$, there exists a number n such that every k-coloring of $\{1, \ldots, n\}$ contains a monochromatic length ℓ arithmetic progression.

- Rephrased: there is a positive integer d such that there is a monochromatic solution to the system of equations

$$x_2 = x_1 + d,\ x_3 = x_2 + d, \ldots,\ x_\ell = x_{\ell-1} + d.$$

- Originally conjectured by Schur while studying quadratic residues.

Van der Waerden numbers

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Example: $w(2, 3) = 9$

1 2 3 4 5 6 7 8
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Van der Waerden numbers

Best general bounds:

- $w(2, p+1) \geq p2^p$ (Berlekamp '68)
- $w(k, \ell) \leq 2^{2^{k^{2^{2^{\ell+9}}}}}$ (Gowers '01)

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All known exact values of $w(k, \ell)$:

ℓ \ k	2	3	4
3	9	27	76
4	35	293	
5	178		
6	1132		

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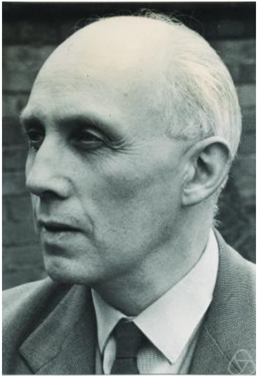
Answered by Rado in his PhD thesis with Schur.



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- The Schur number $S(k)$ is $R_k(x + y = z)$.
- Not all Rado numbers are finite: e.g., $R_2(2x + 2y = z) = 34$, $R_3(2x + 2y = z) = \infty$.

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Rado's theorem generalizes Schur's theorem:

Theorem (Rado's Single Equation Theorem, 1933)

Let $m \geq 2$, and let $c_i \in \mathbb{Z} \setminus \{0\}$. Then the equation

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- $4w - 2x + 3y - 7z = 0$ is regular.
- $2x - 5y + 6z = 0$ is NOT regular.

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- $S(5) = 161$ (Heule 2018)
- Generalized 3-color Schur numbers (Boza-Marín-Revuelta-Sanz 2019)

Schur numbers

Table of all known Schur numbers:

k	$S(k)$	
1	2	
2	5	
3	14	
4	45	Golomb and Baumert '65
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- Our goal: study Rado numbers using SAT solvers

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Theorem (Chang-De Loera-W '22)

1. $R_2(ax + by = cz)$ *for* $1 \leq a, b, c \leq 20$.
2. $R_3(a(x - y) = bz)$ *for* $1 \leq a, b \leq 15$.
3. $R_3(a(x + y) = bz)$ *for* $1 \leq a, b \leq 10$.
4. $R_3(ax + by = cz)$ *for* $1 \leq a, b, c \leq 6$.
5. $R_4(x - y = az)$ *for* $1 \leq a \leq 4$.
6. $R_4(a(x - y) = z)$ *for* $1 \leq a \leq 5$.

Patterns in $R_3(a(x-y)=bz)$

b \ a	1	2	3	4	5	6	7	8	9	10	11	12
1	14	14	27	64	125	216	343	512	729	1000	1331	1728
2	43	*14*	31	*14*	125	*27*	343	*64*	729	*125*	1331	*216*
3	94	61	*14*	73	125	*14*	343	512	*27*	1000	1331	*64*
4	173	*43*	109	*14*	141	*31*	343	*14*	729	*125*	1331	*27*
5	286	181	186	180	*14*	241	343	512	729	*14*	1331	1728
6	439	*94*	*43*	*61*	300	*14*	379	*73*	*31*	*125*	1331	*14*
7	638	428	442	456	470	462	*14*	561	729	1000	1331	1728
8	889	*173*	633	*43*	665	*109*	644	*14*	793	*141*	1331	*31*
9	1198	856	*94*	892	910	*61*	896	896	*14*	1081	1331	*73*
10	1571	*286*	1171	*181*	*43*	*186*	1190	*180*	1206	*14*	1431	*241*
11	2014	1508	1530	1552	1574	1596	1618	1584	1575	1580	*14*	1849
12	2533	*439*	*173*	*94*	2005	*43*	2053	*61*	*109*	*300*	2024	*14*

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- $R_3(a(x-y)=bz)=a^3$ for $a \geq b+2$, $\gcd(a,b)=1$
- $R_3(a(x-y)=(a-1)z)=a^3+(a-1)^2$
- $R_3(x-y=bz)=(b+2)^3-(b+2)^2-(b+2)-1$

Patterns in $R_3(a(x-y)=bz)$

Theorem (Chang-De Loera-W '22)

- $R_3(a(x-y)=bz)=a^3$ for $a \geq b+2$, $a \geq 3$, $\gcd(a,b)=1$,
- $R_3(a(x-y)=(a-1)z)=a^3+(a-1)^2$ for $a \geq 3$,
- $R_3(x-y=bz)=(b+2)^3-(b+2)^2-(b+2)-1$ for $b \geq 1$.

Patterns in $R_3(a(x-y)=bz)$

Theorem (Chang-De Loera-W '22)

- $R_3(a(x-y)=bz)=a^3$ *for* $a \geq b+2$, $a \geq 3$, $\gcd(a,b)=1$,
- $R_3(a(x-y)=(a-1)z)=a^3+(a-1)^2$ *for* $a \geq 3$,
- $R_3(x-y=bz)=(b+2)^3-(b+2)^2-(b+2)-1$ *for* $b \geq 1$.

This theorem implies that the following result, conjectured by Ahmed and Schaal in 2016, on the *generalized Schur numbers* $S(k,m) := R_k(x_1+\cdots+x_{m-1}=x_m)$ is true:

Theorem (Boza,Marín,Revuelta,Sanz '19)

$S(3,m)=m^3-m^2-m-1.$

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- $F_n^k(\mathcal{E})$ has three types of clauses: positive, negative, and optional

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- Positive clauses encode that each integer is assigned at least one color, and take the form

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- Negative clauses encode that there are no monochromatic solutions to $\mathcal{E}$. For each solution (x, y, z) and color c, we have the negative clause

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- Optional clauses encode that each integer is assigned at most one color, and take the form

$$\bar{v}_i^c \vee \bar{v}_i^{c'}$$

  for all i and all colors $1 \leq c < c' \leq k$.

The clauses in the formula $F_4^3(x+y=z)$ are:
Positive clauses:

$$(v_1^1 \vee v_1^2 \vee v_1^3) \wedge (v_2^1 \vee v_2^2 \vee v_2^3) \wedge (v_3^1 \vee v_3^2 \vee v_3^3) \wedge (v_4^1 \vee v_4^2 \vee v_4^3)$$

Negative clauses:

$$(\overline{v}_1^1 \vee \overline{v}_1^1 \vee \overline{v}_2^1) \wedge (\overline{v}_2^1 \vee \overline{v}_1^1 \vee \overline{v}_3^1) \wedge (\overline{v}_3^1 \vee \overline{v}_1^1 \vee \overline{v}_4^1) \wedge$$
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- Four color formulas require symmetry breaking and more time

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- For each equation $\mathcal{E}$ we produced the satisfiable formula $F^k_{n-1}(\mathcal{E})$ and unsatisfiable formula $F^k_n(\mathcal{E})$ to prove $R_k(\mathcal{E}) = n$
- Guess lower and upper bounds for n, then use binary search to determine n
- Solving is relatively fast for two or three colors (≤ 1 min for most formulas)
- Four color formulas require symmetry breaking and more time
- Modified encoding to compute **infinitely** many values with a single formula

SAT solving

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- Want to describe sets S that work for an *entire family* of equations

2-color Generalized Schur Numbers

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- $(1,a,a^2-2a+1)$ and $(a-1,a,a^2-a-1)$ are solutions, so a cannot be either color.

Upper bounds: Variables indexed by polynomials

- Can prove upper bounds by generating a formula in the same way: negative clauses look like

$$\overline{v_1}^1 \vee \overline{v_a}^1 \vee \overline{v}^1_{a^2-2a+1}.$$

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- Set S describes an unsatisfiable core for an equation
- Found suitable S for our Rado number families:

$\mathcal{E}$	$R_3(\mathcal{E})$	$\lvert S \rvert$
$x - y = (b-2)z$	$b^3 - b^2 - b - 1$	685
$a(x-y) = (a-1)z$	$a^3 + (a-1)^2$	1365
$a(x-y) = bz$	a^3	40645

Current work

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- Application to other families of Rado numbers (nonhomogeneous, nonlinear, etc.)
- General formula for $R_2(ax + by = cz)$

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- Fox and Kleitman proved $\Delta = 24$ suffices for $m = 3$, but unknown if this can be improved
- Is there an equation $\mathcal{E}$ of the form $ax + by + cz = 0$ with degree of regularity 4?

Polynomial encodings

Given an equation $\mathcal{E}$ and number of colors k, let S_n be the set of solutions to $\mathcal{E}$ where each variable is in $\{1, \ldots, n\}$.
The following system of equations has no solution over $\overline{\mathbb{F}_2}$ if and only if $R_k(\mathcal{E}) \leq n$:

$$
\begin{aligned}
\prod_{s \in S_n} x_{s,c} &= 0 && 1 \leq c \leq k, \\
1 + \sum_{c=1}^{k} x_{i,c} &= 0 && 1 \leq i \leq n, \\
x_{i,c} x_{i,c'} &= 0 && 1 \leq c < c' \leq k.
\end{aligned}
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1 + x_i + y_i &= 0, & 1 \leq i &\leq 5.
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Hilbert's Nullstellensatz

Theorem (Hilbert, 1893)

Let K be an algebraically closed field, and let $f_1, \dots, f_m \in K[x_1, \dots, x_n]$. Then there is no solution to the system $f_1 = \cdots = f_m = 0$ if and only if there exist polynomials $\beta_1, \dots, \beta_m$ such that $\sum_{i=1}^{m} \beta_i f_i = 1$.

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Goal: describe Nullstellensatz certificates for Rado numbers

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Consider the following game between the two players **Builder** and **Painter**:

- Fix an equation $\mathcal{E}$ and positive integers k and n.

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- Builder always wins if $n \geq R_k(\mathcal{E})$.
- Example: $\tilde{R}_2(x + 3y = 3z; 9) \leq 5$ (Builder can choose from $\{3, 4, 6, 7, 9\}$ and win)

Online Ramsey numbers

Theorem (De Loera-W)

Using the previous encoding, there exists a Nullstellensatz certificate of degree at most $\tilde{R}_k(\mathcal{E}; n)$ *for* $n = R_k(\mathcal{E})$.

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This theorem and encoding generalize!

- Ramsey numbers (multicolor, arbitrary graphs)
- Schur and Rado numbers
- van der Waerden numbers
- Hales-Jewett numbers

Future work

- Find lower bounds for the degrees of Nullstellensatz certificates in this encoding. Are online Ramsey-type numbers good bounds?
- The inequalities

  min Nullstellensatz degree $\leq$ online Rado number $\leq$ Rado number

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  are strict in general
- Investigate the analogous Builder-Painter game for other problems (Schur numbers, van der Waerden numbers, Ramsey numbers for other graphs)

Thank you!

References:

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- (with Jesús De Loera) **Ramsey Numbers through the Lenses of Polynomial Ideals and Nullstellensätze** https://arxiv.org/abs/2209.13859