The Proof Complexity of Integer Programming

Noah Fleming

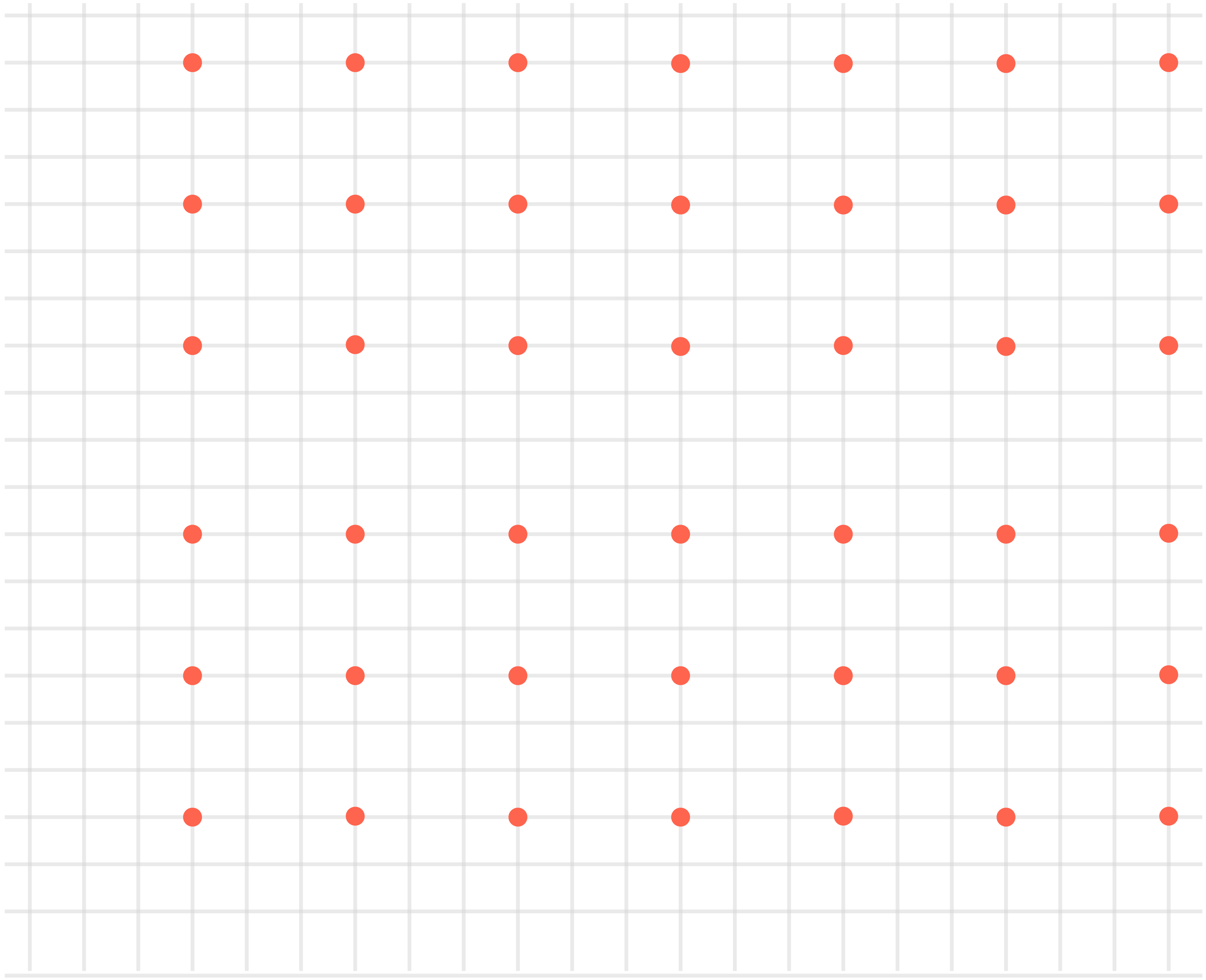
Memorial University

Integer Programming

Input

- A system of linear inequalities

$$a_1 x \geq b_1, \ldots, a_m x \geq b_m$$



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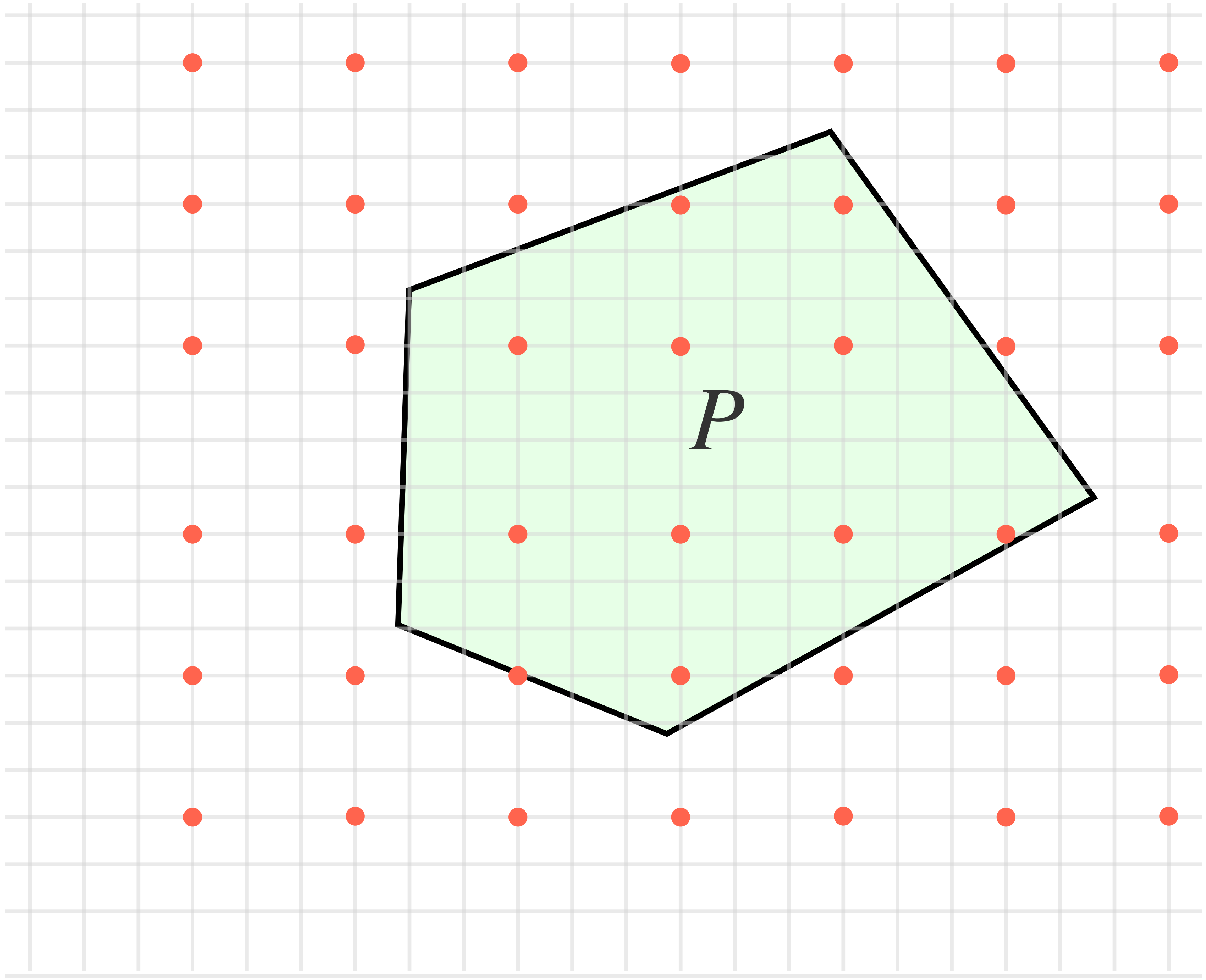

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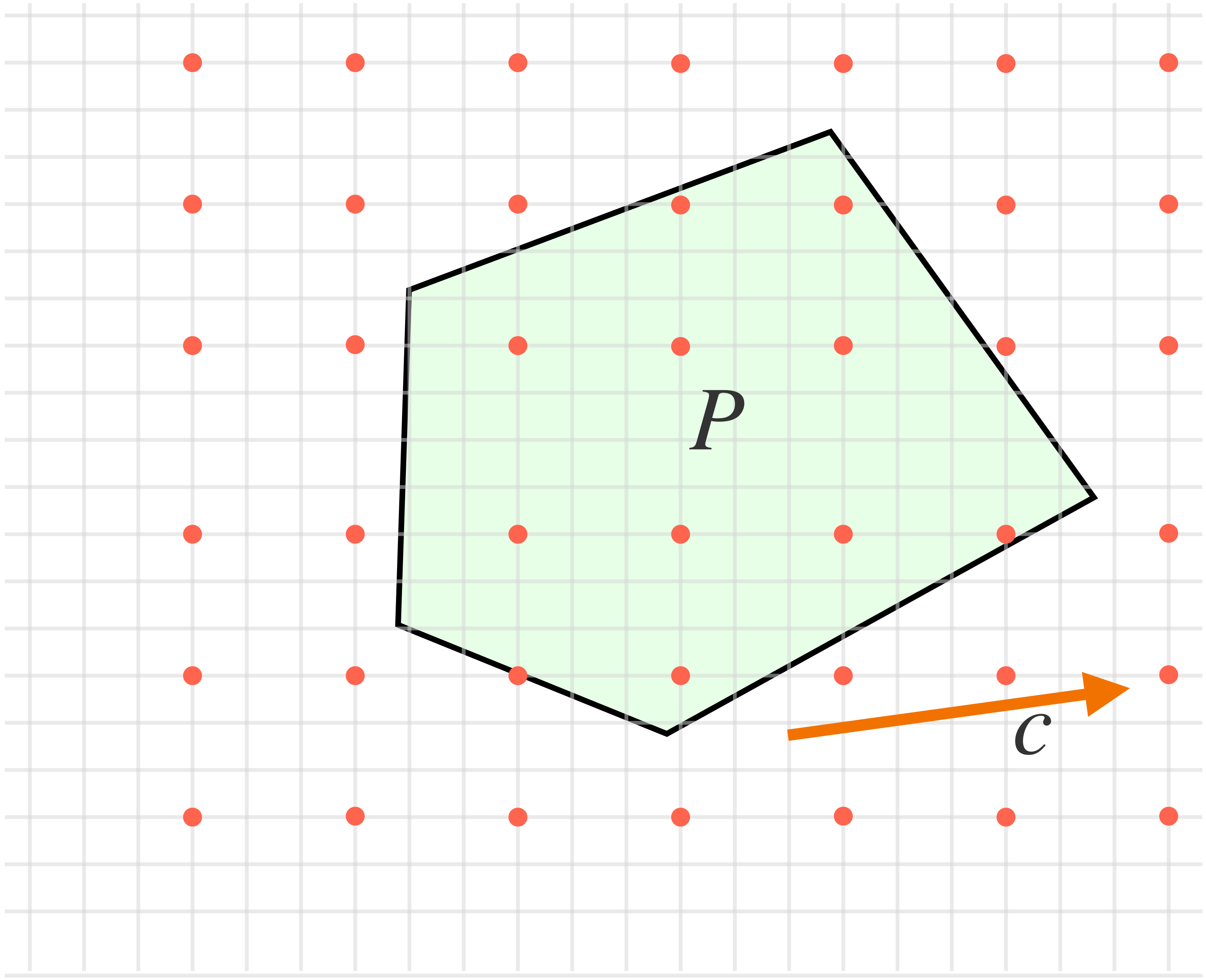

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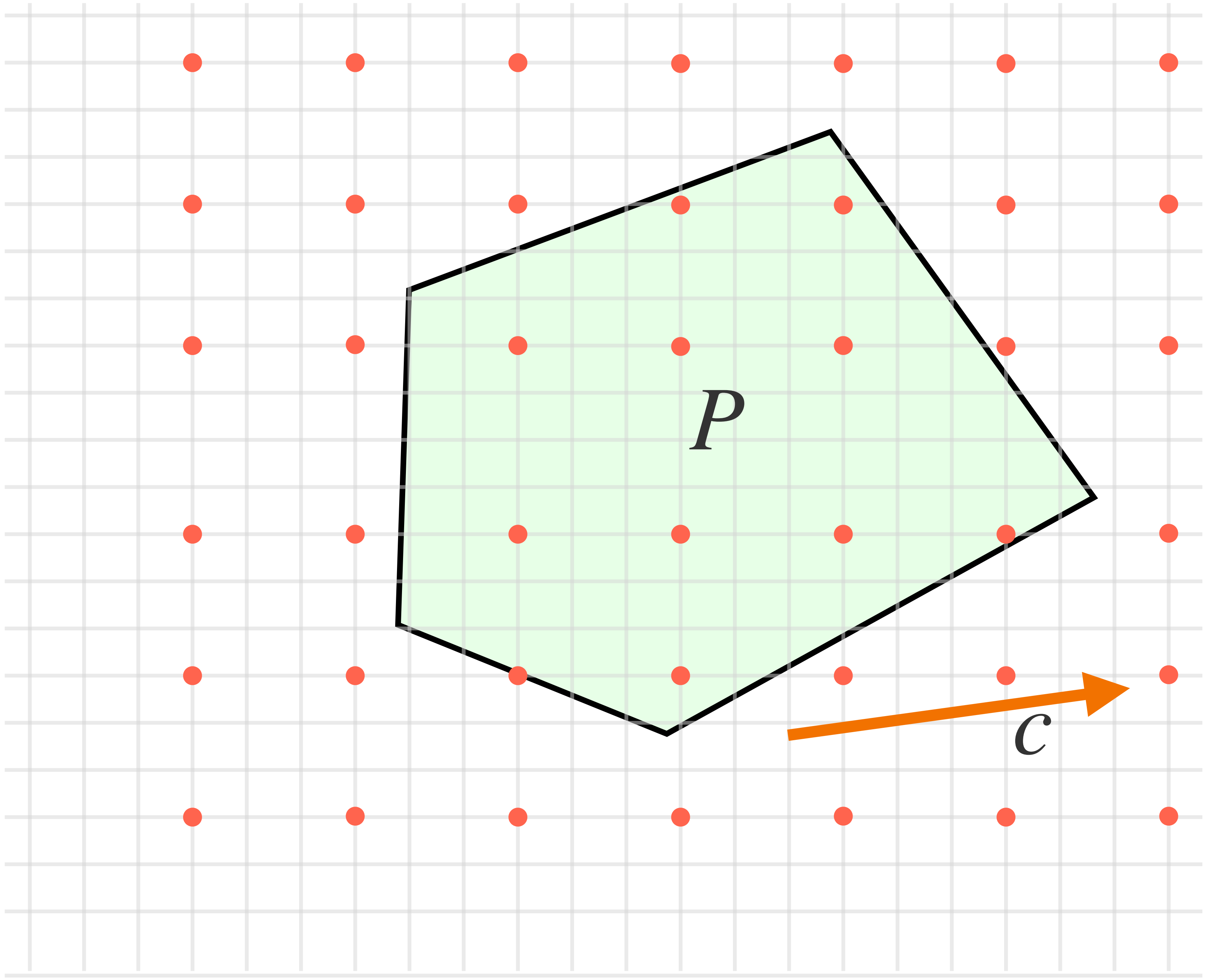

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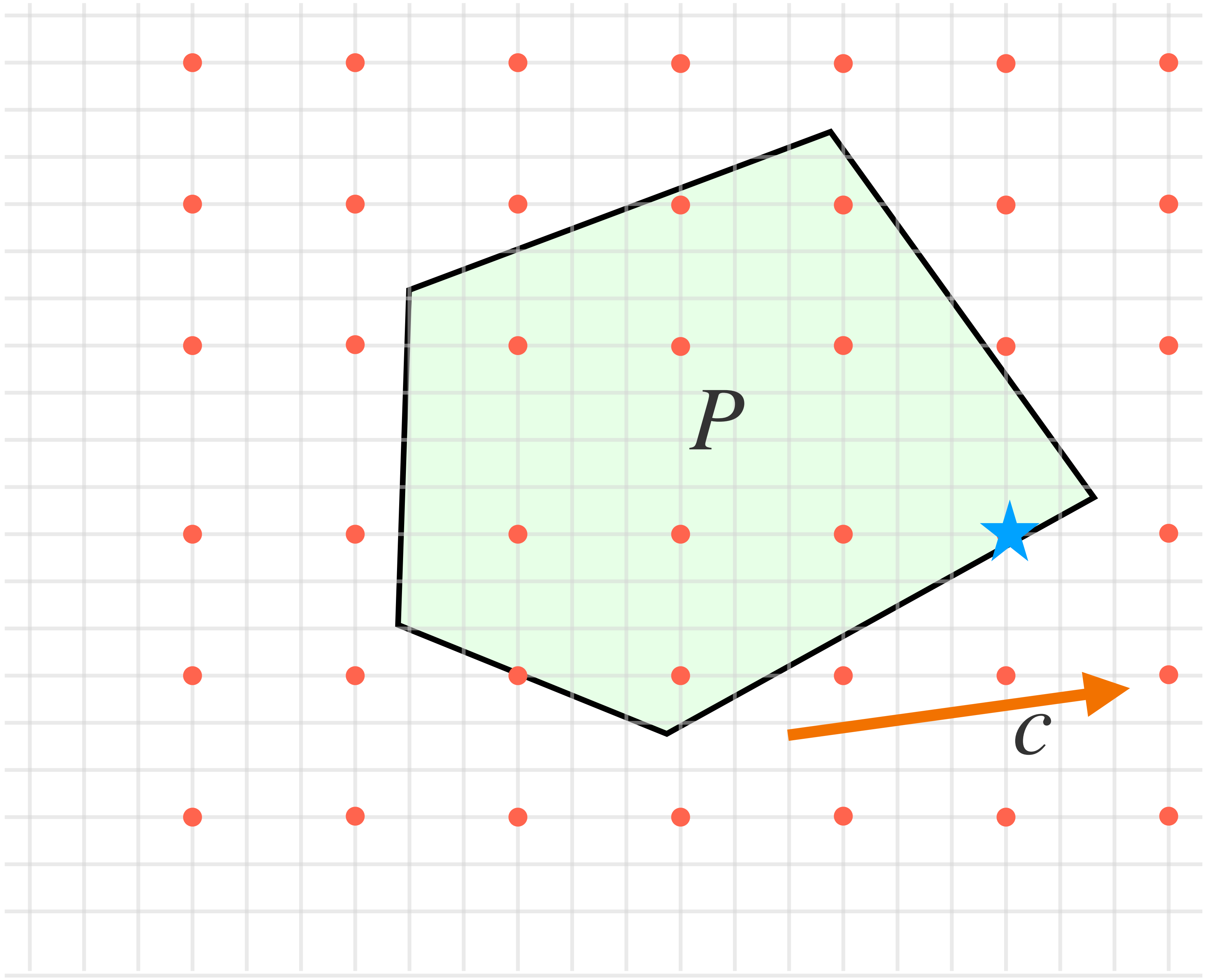

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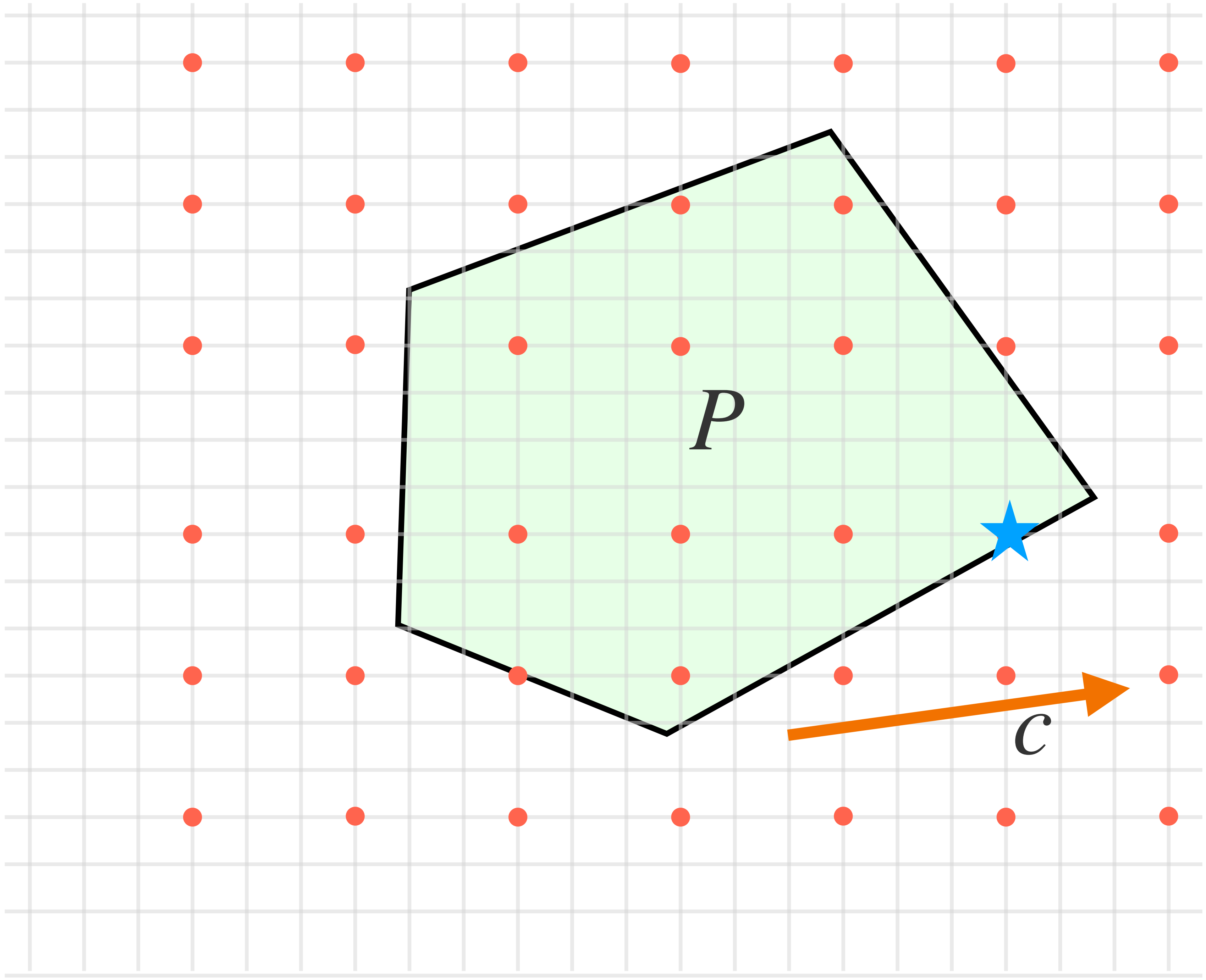

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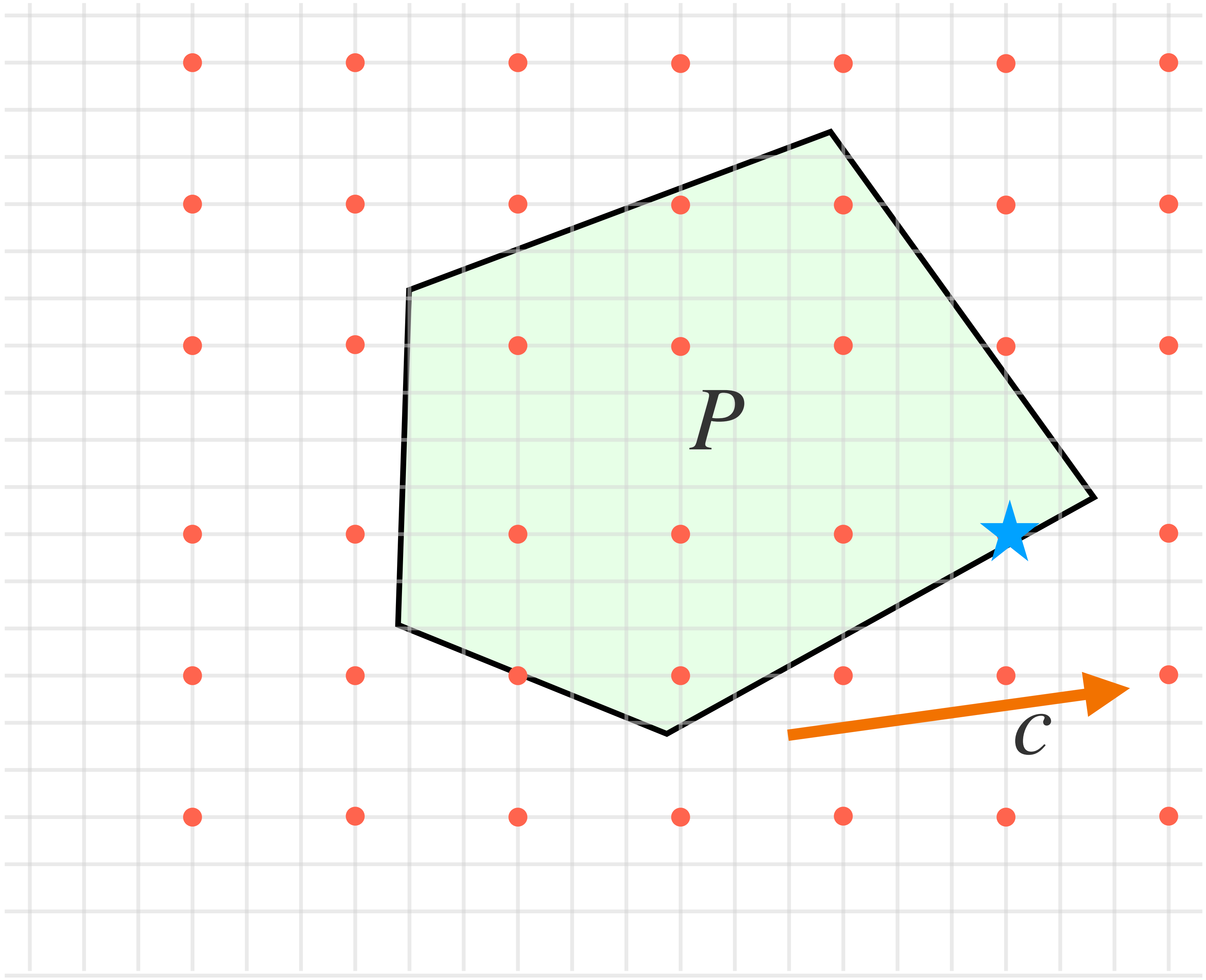

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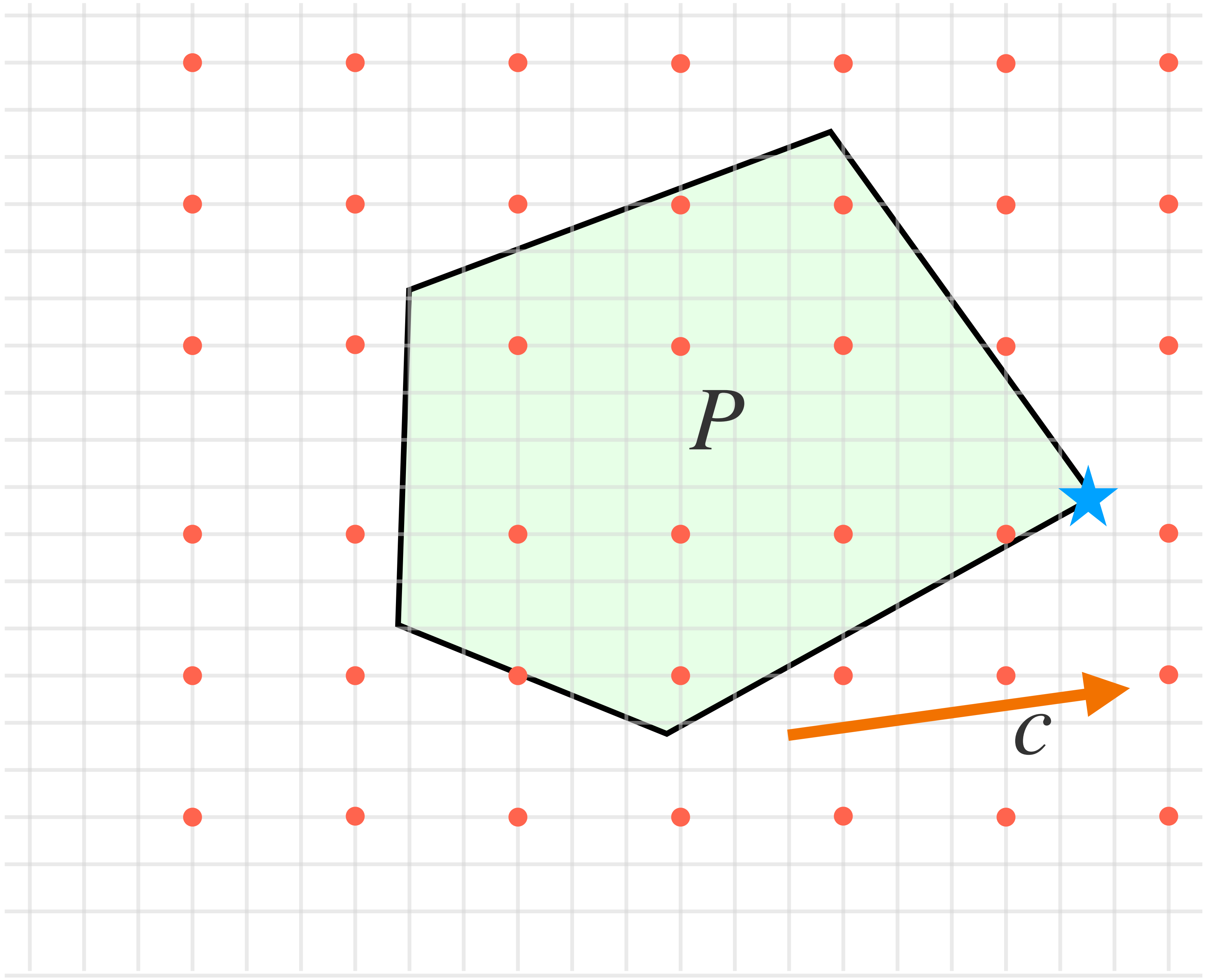

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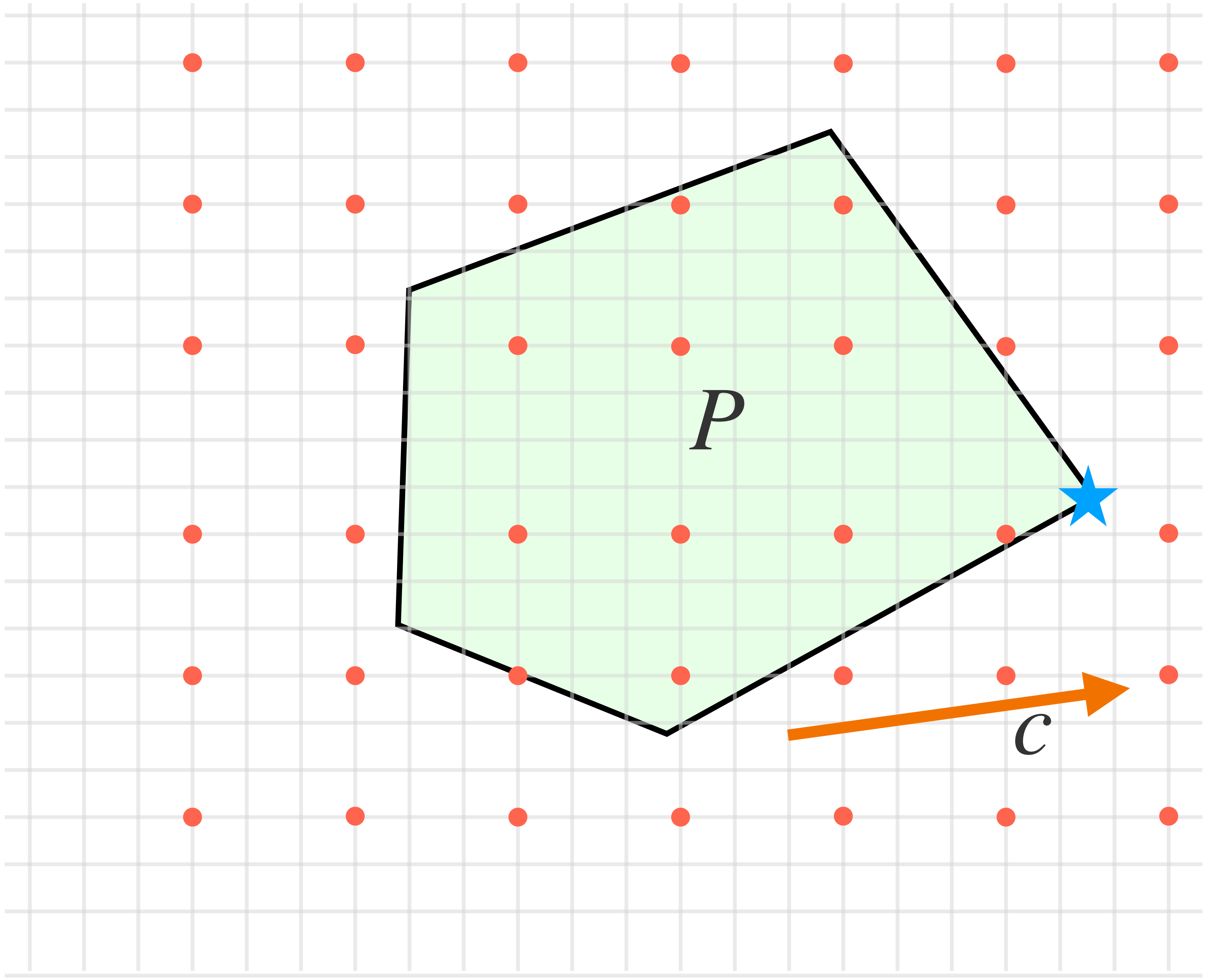

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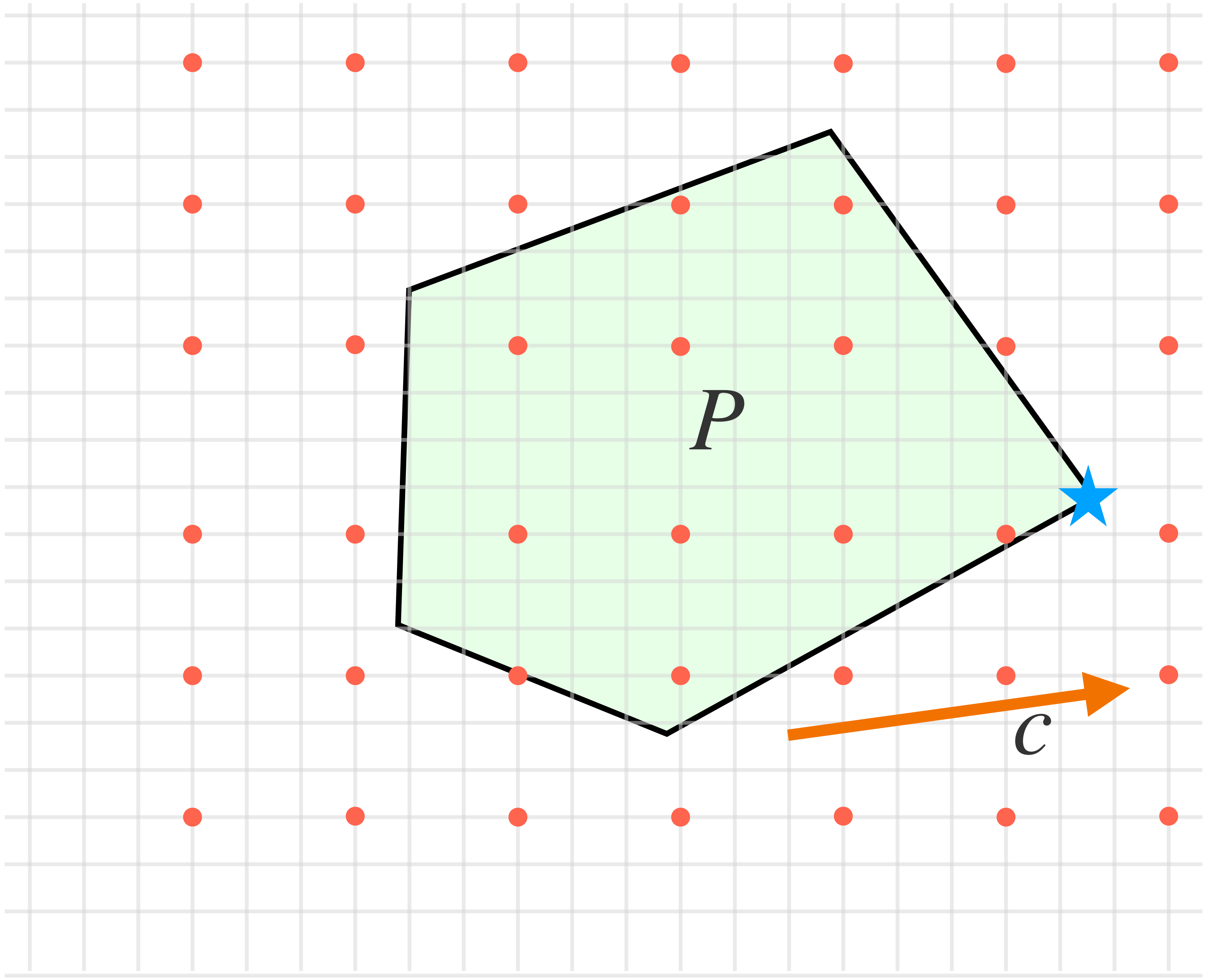

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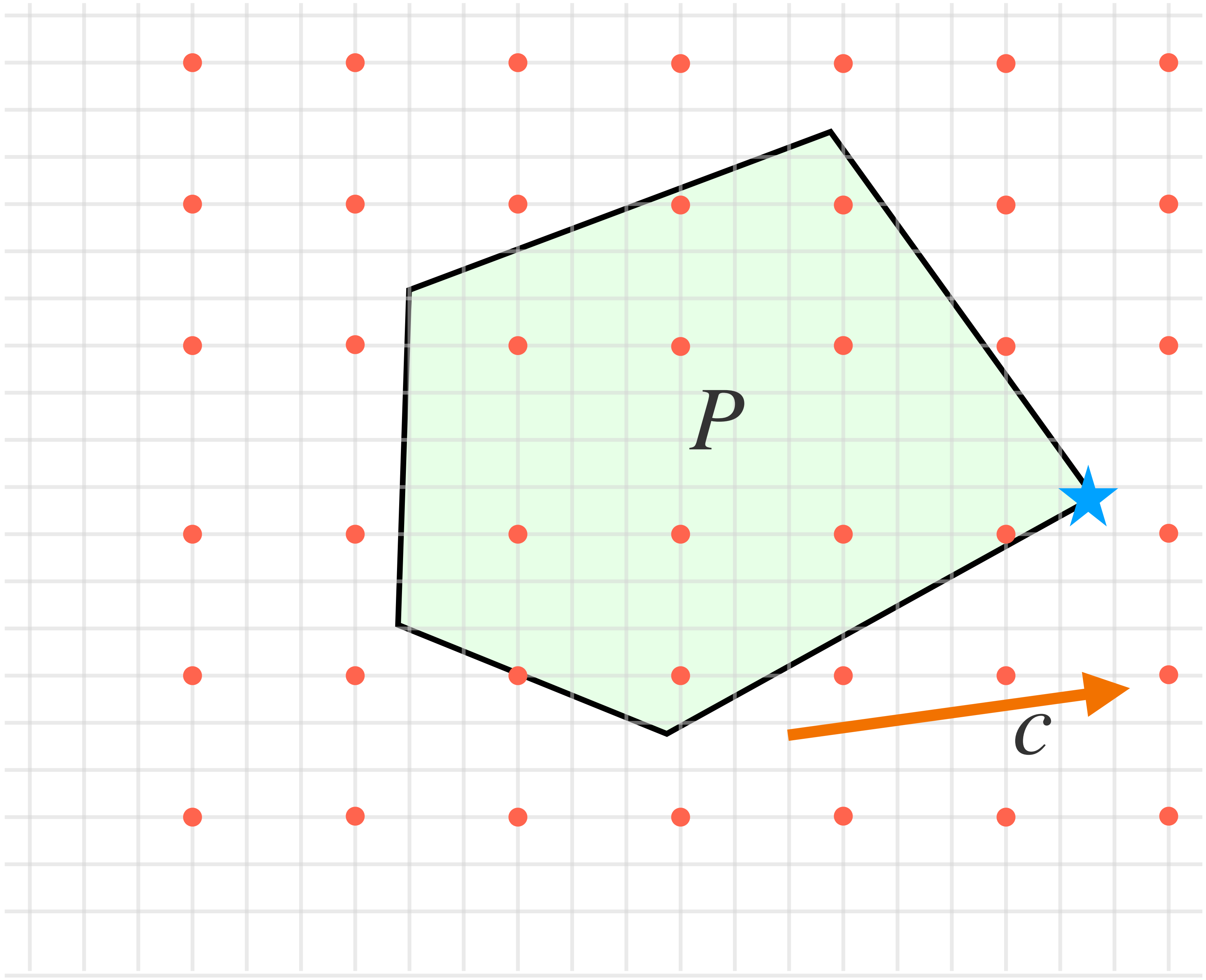

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→ How? — Reduce to linear programming!

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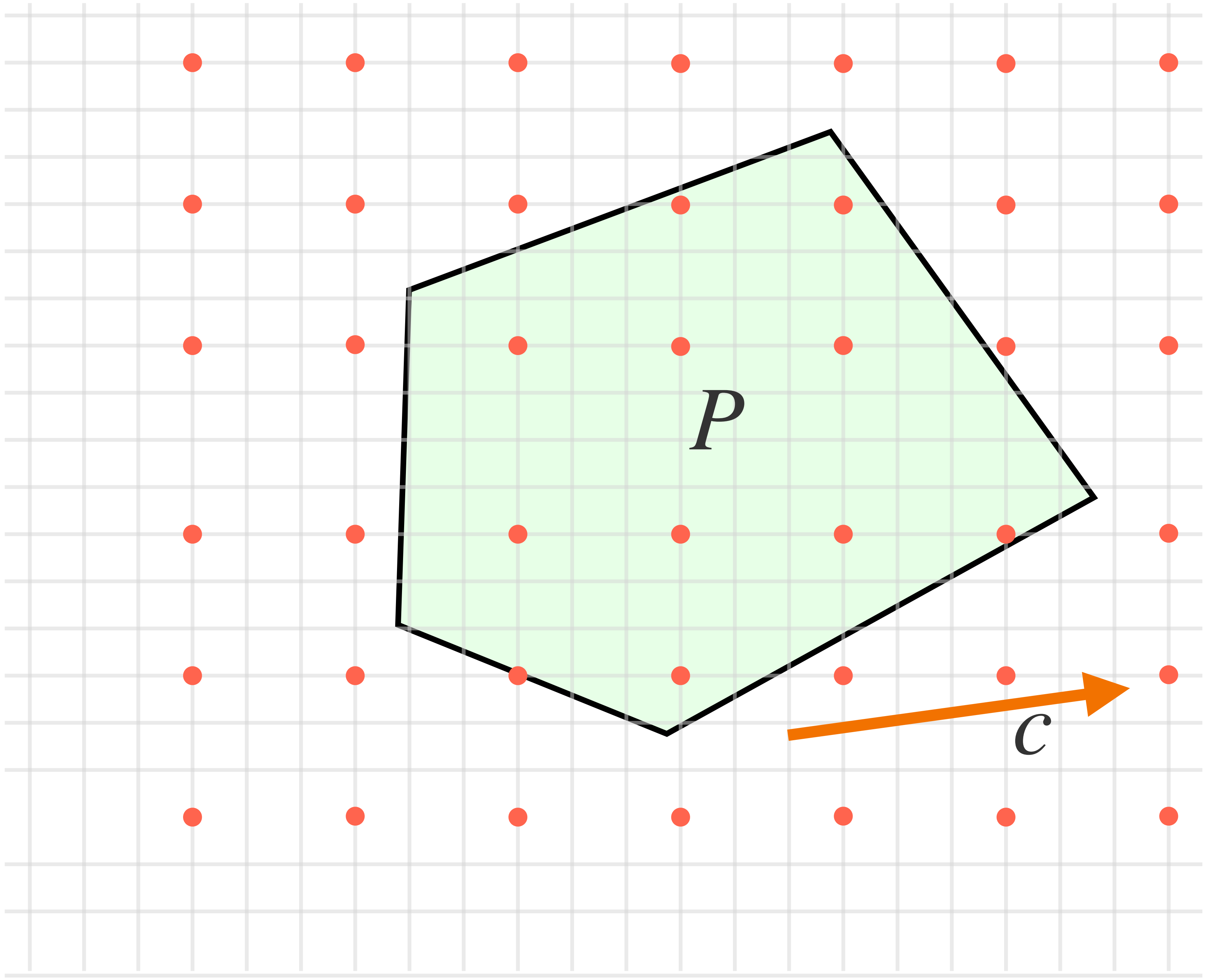
Integer Hull of a polytope P is

$$\text{int}(P) := \text{conv}(P \cap \mathbb{Z}^n)$$



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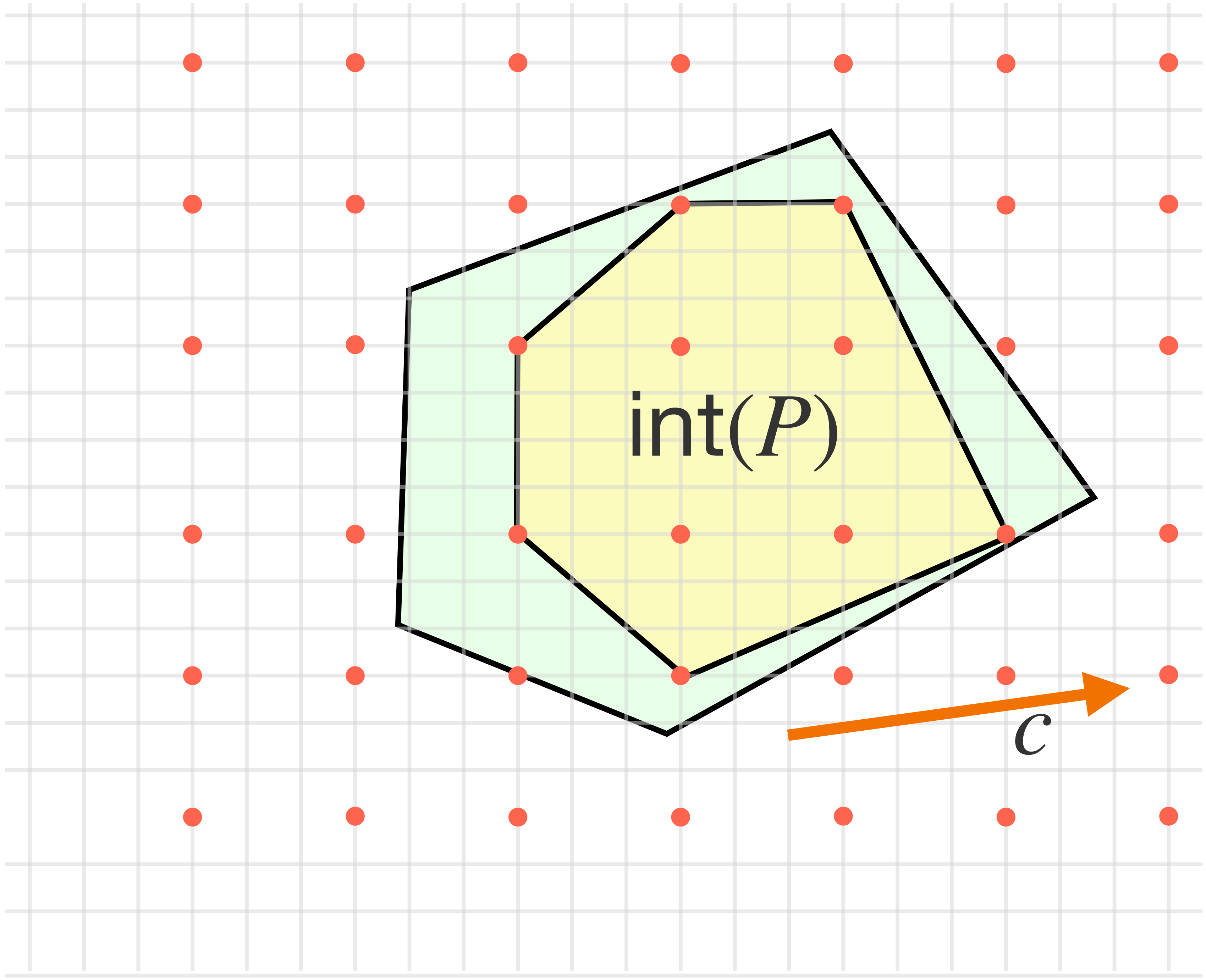

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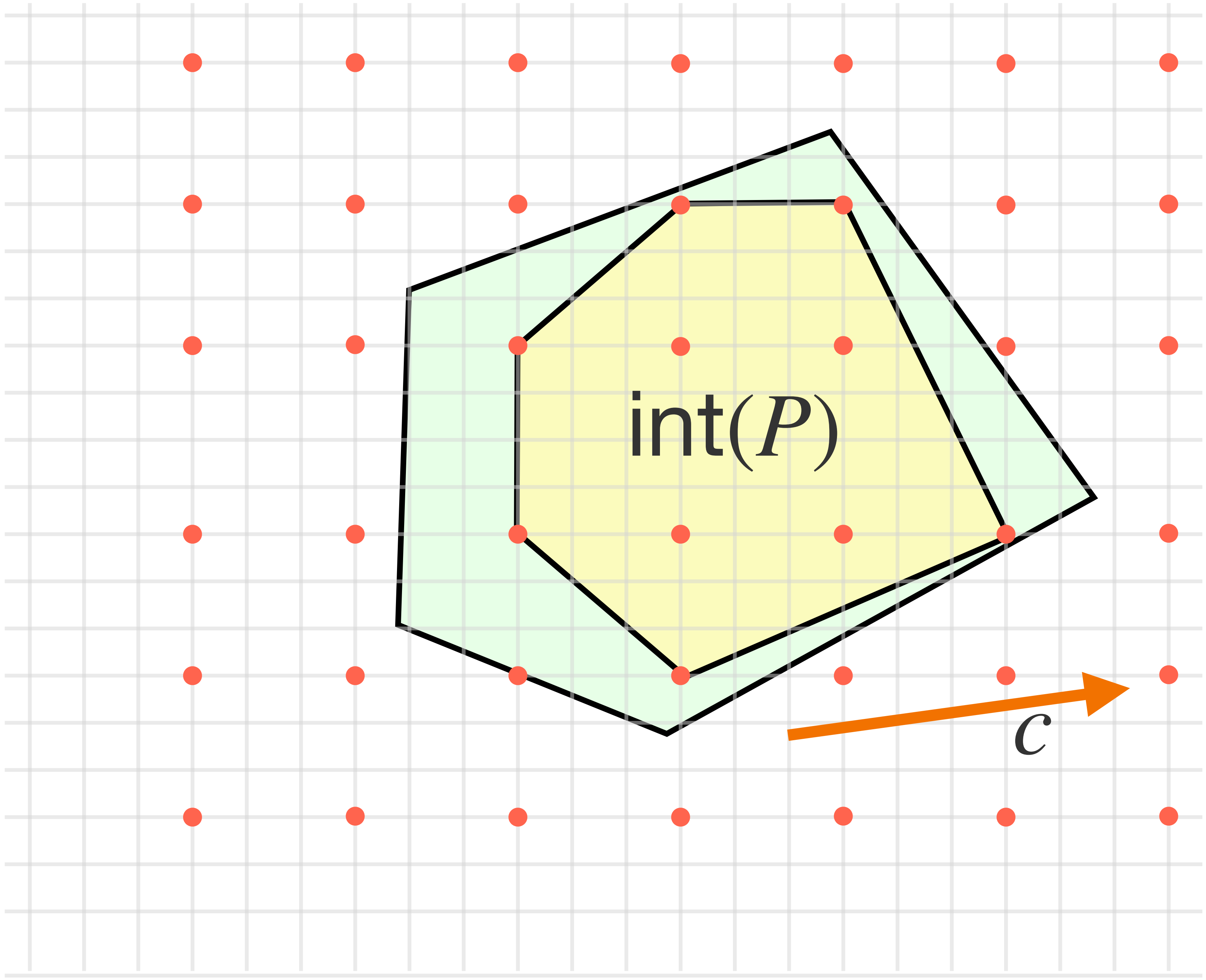

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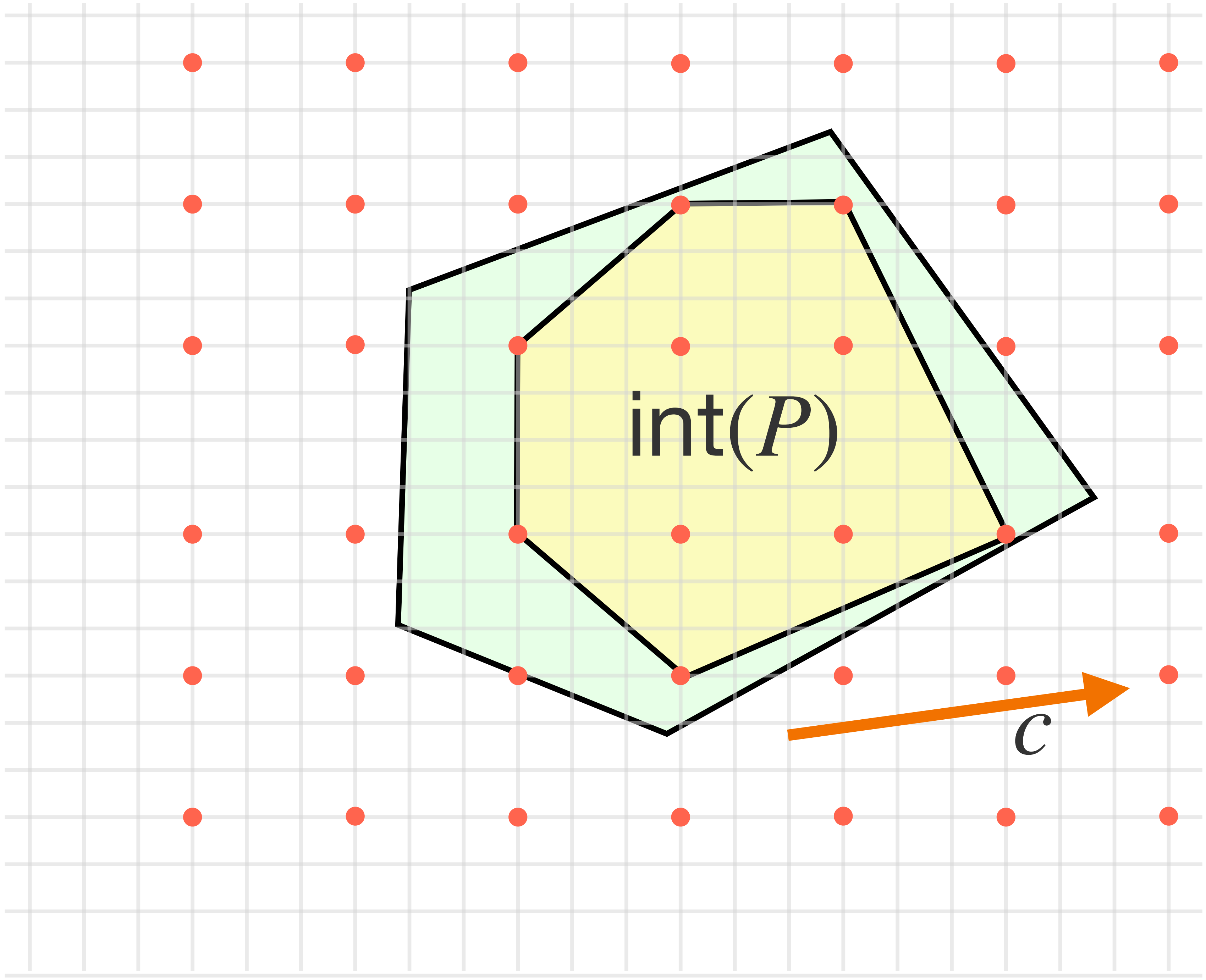

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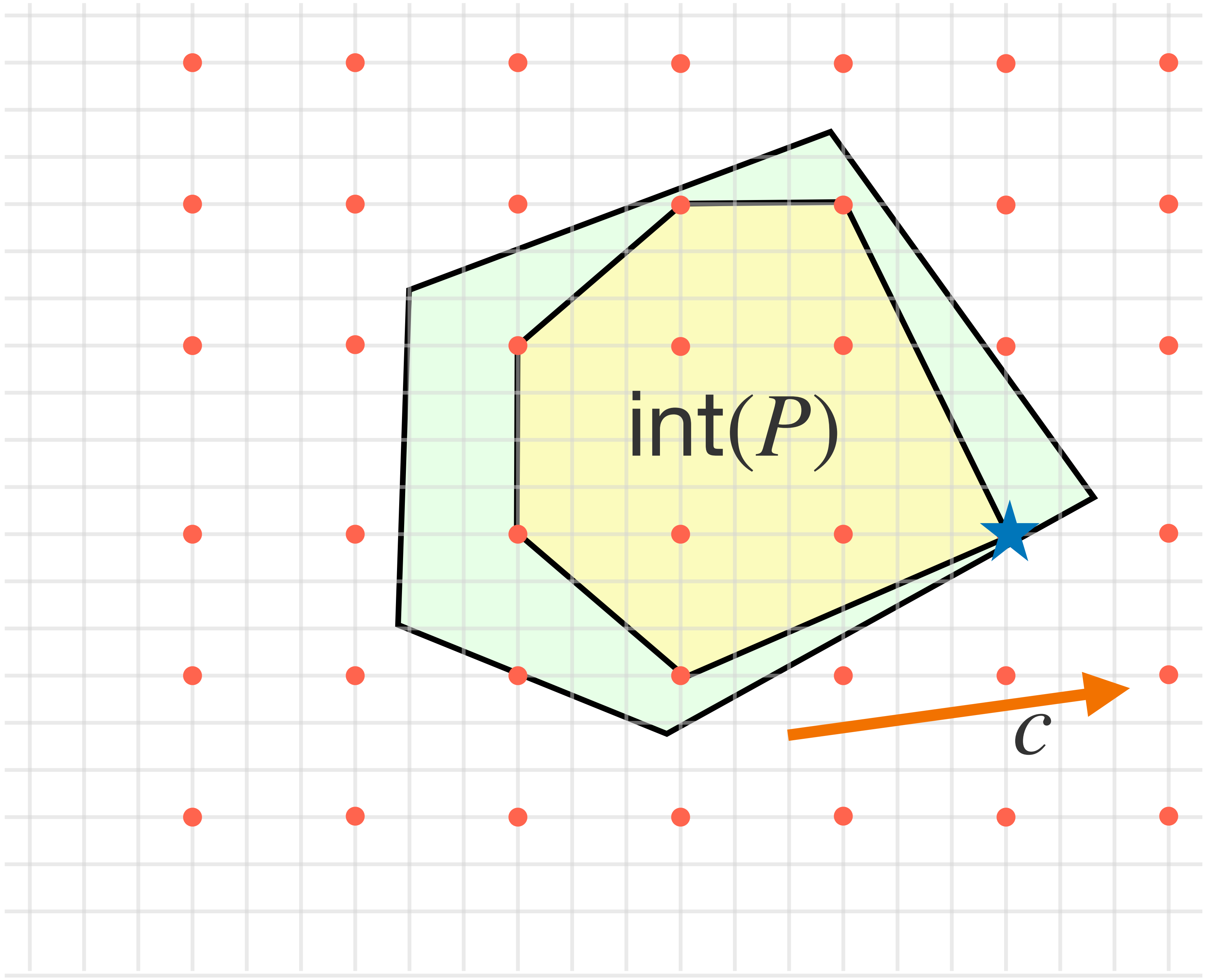

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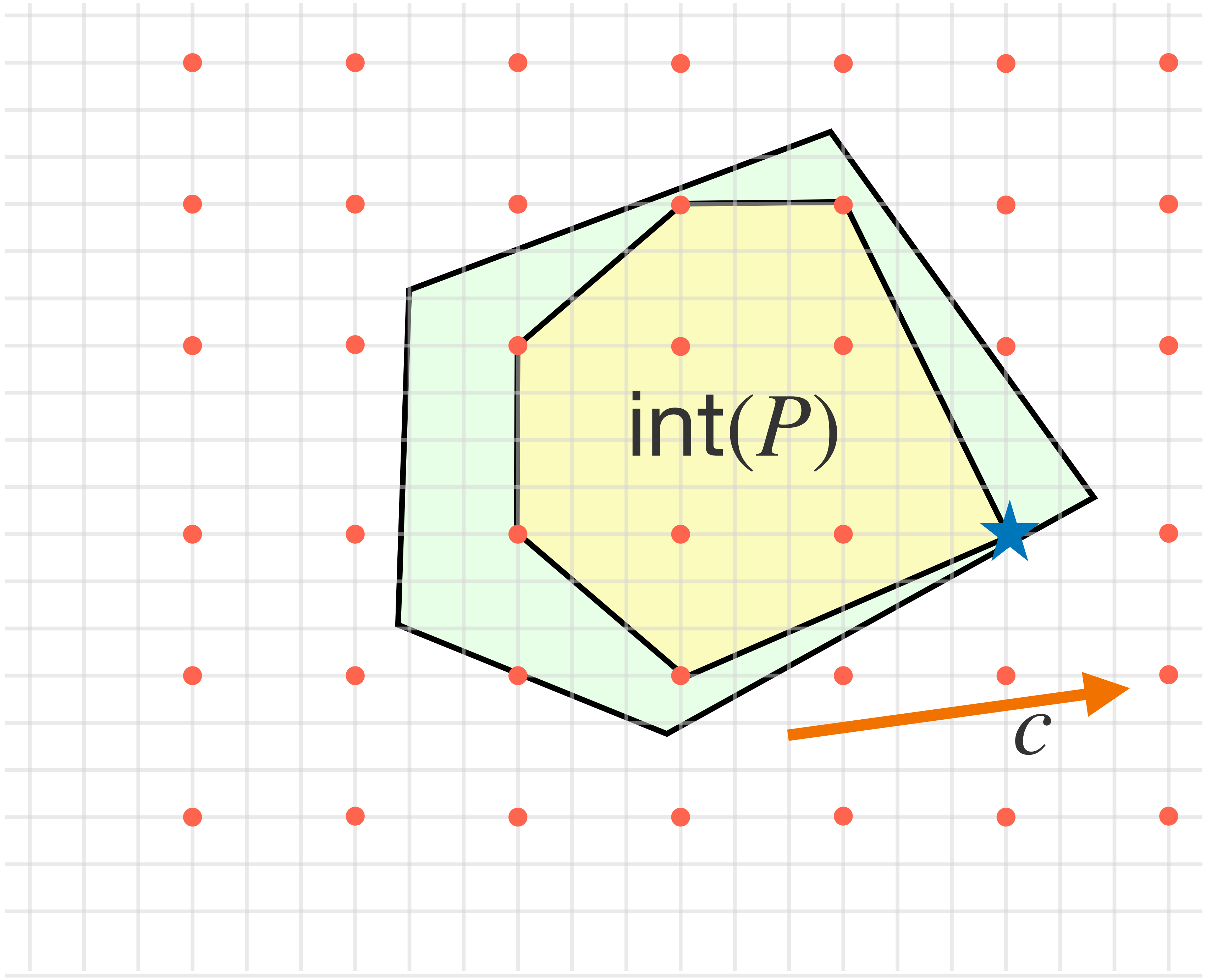

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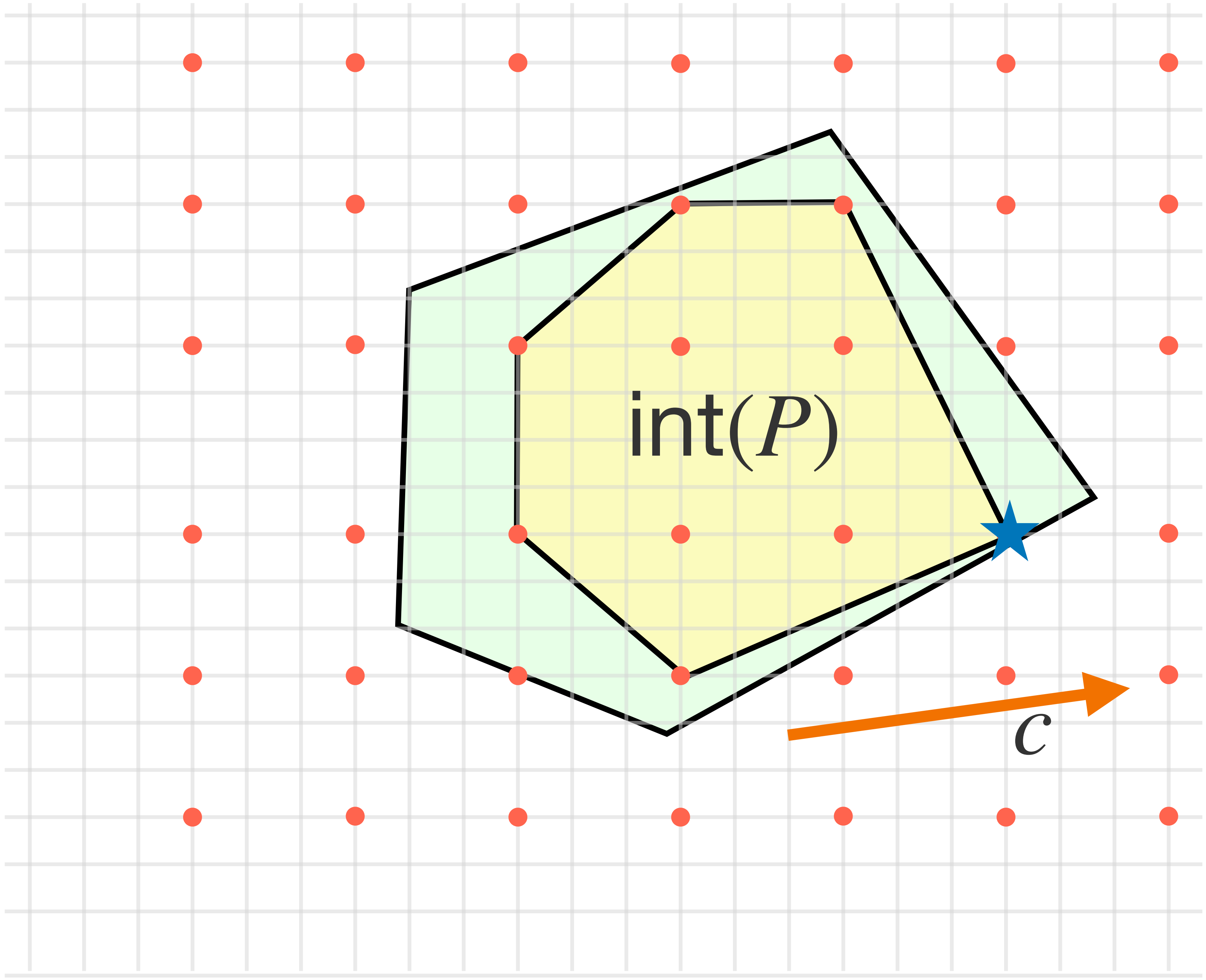

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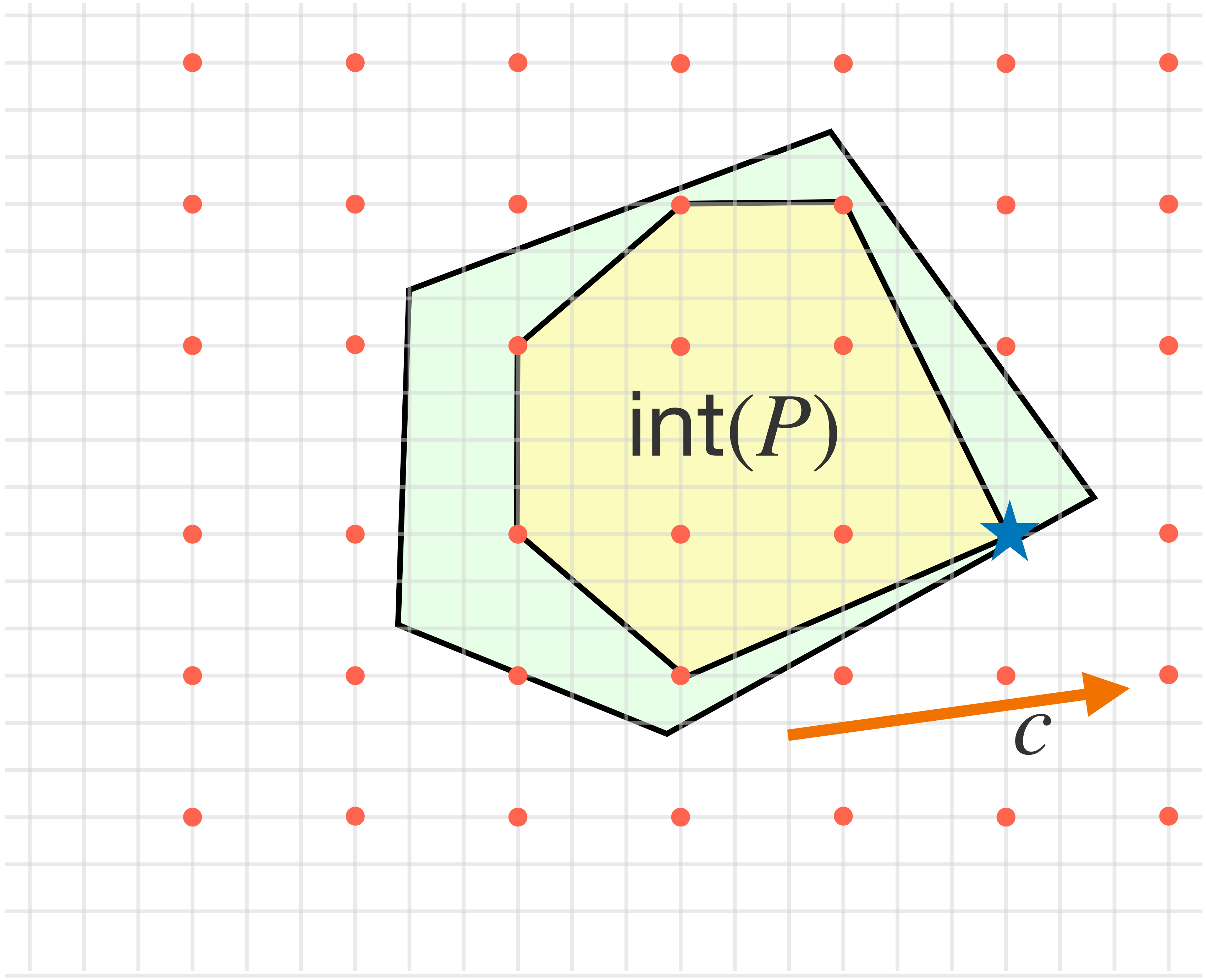

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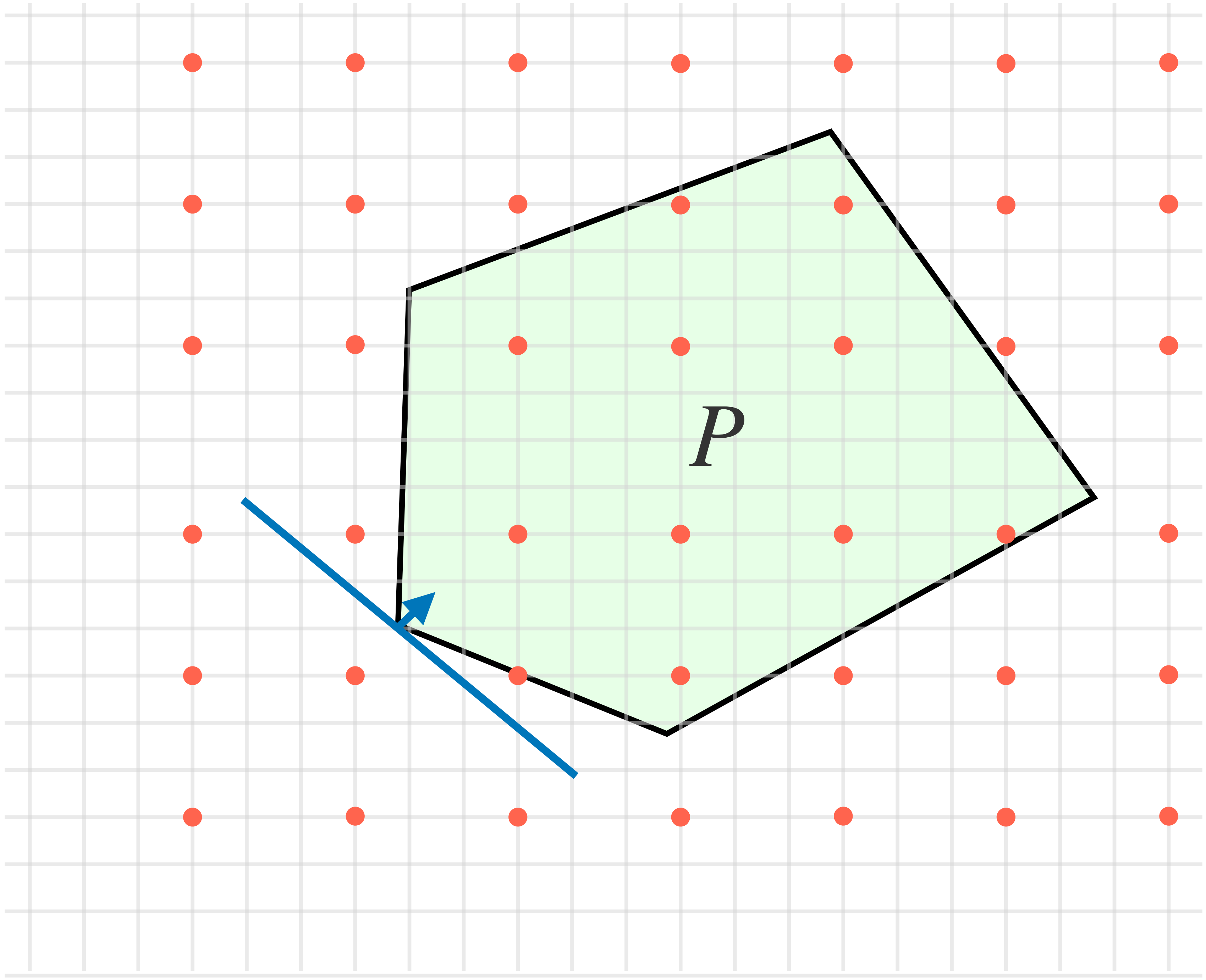

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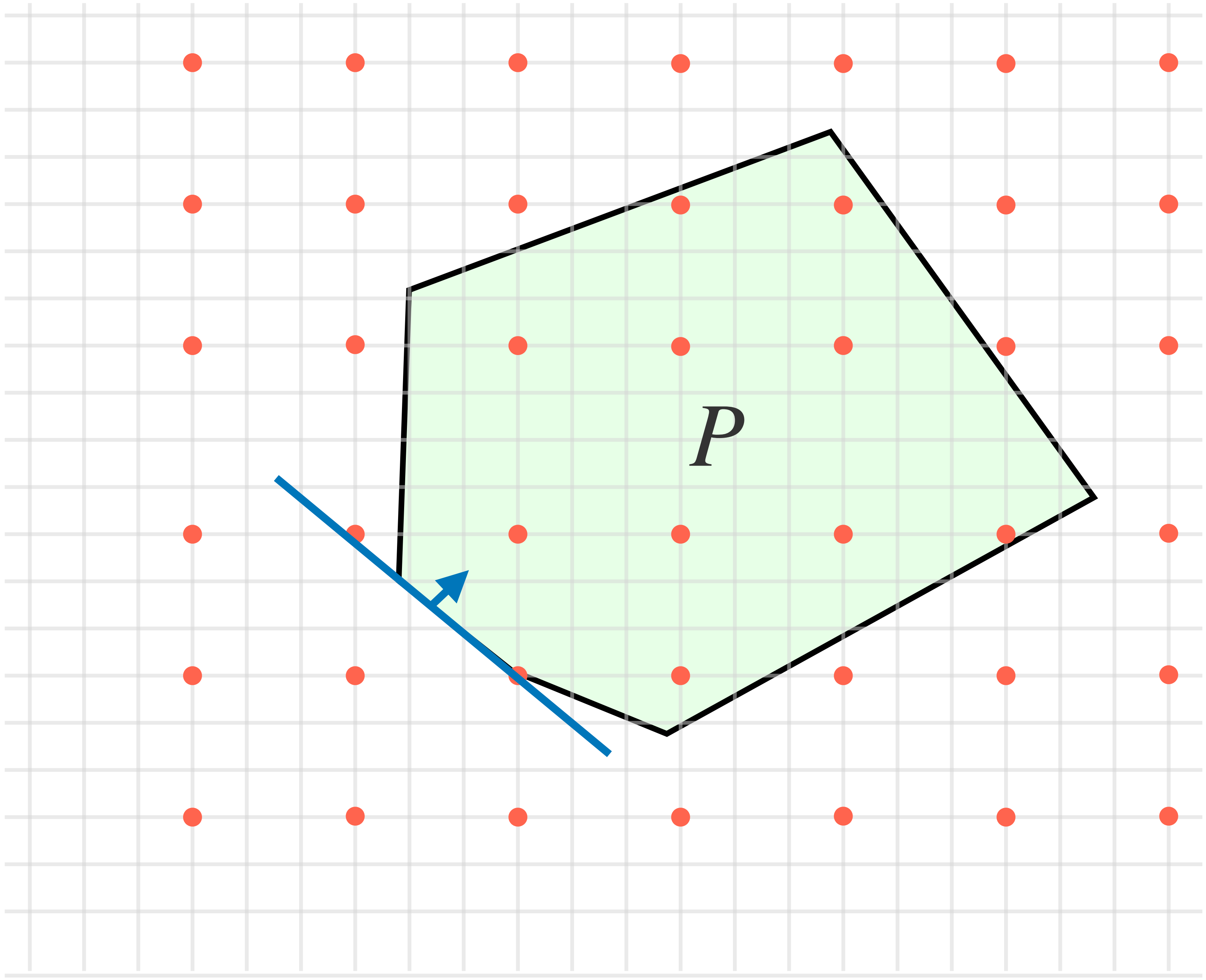

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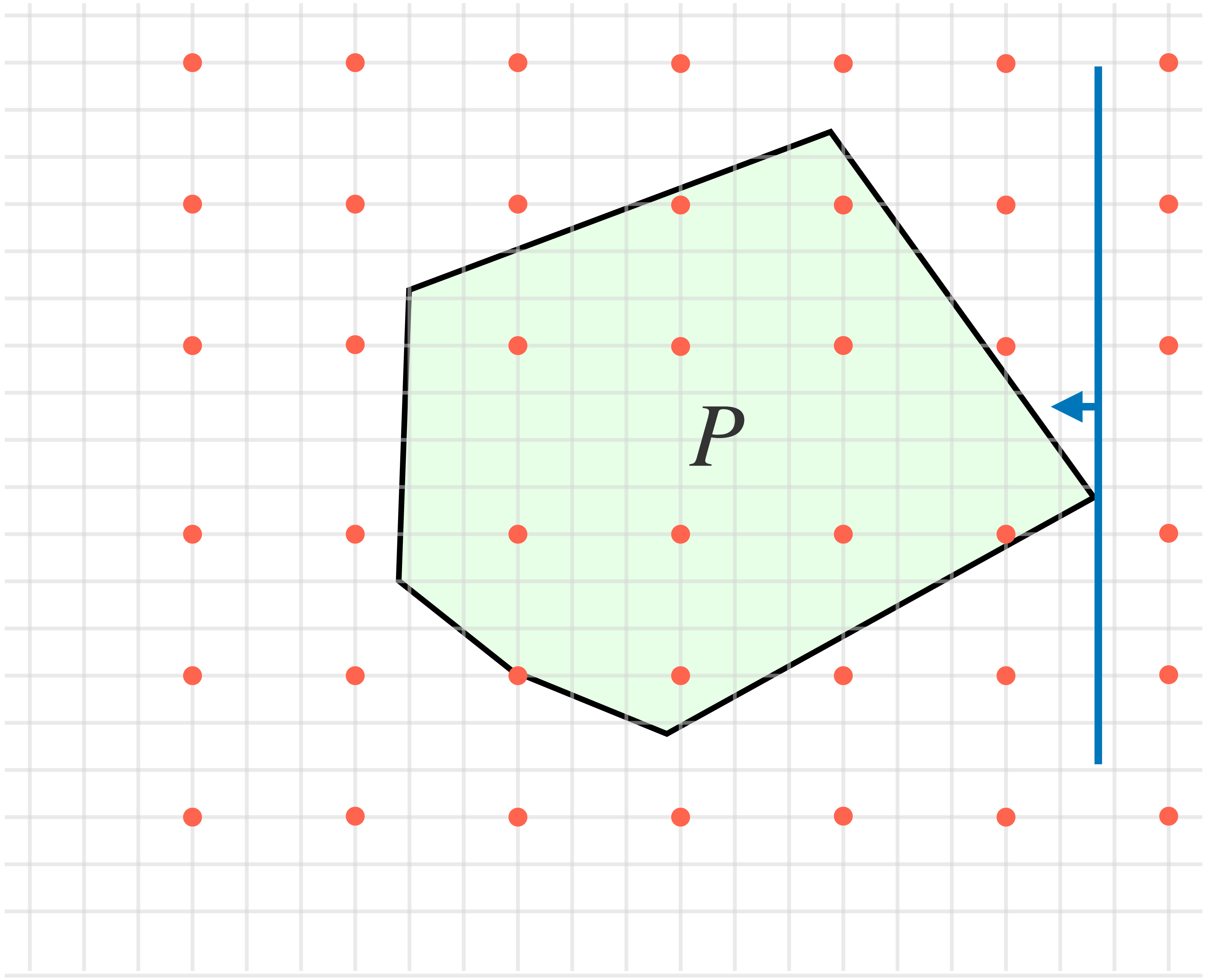

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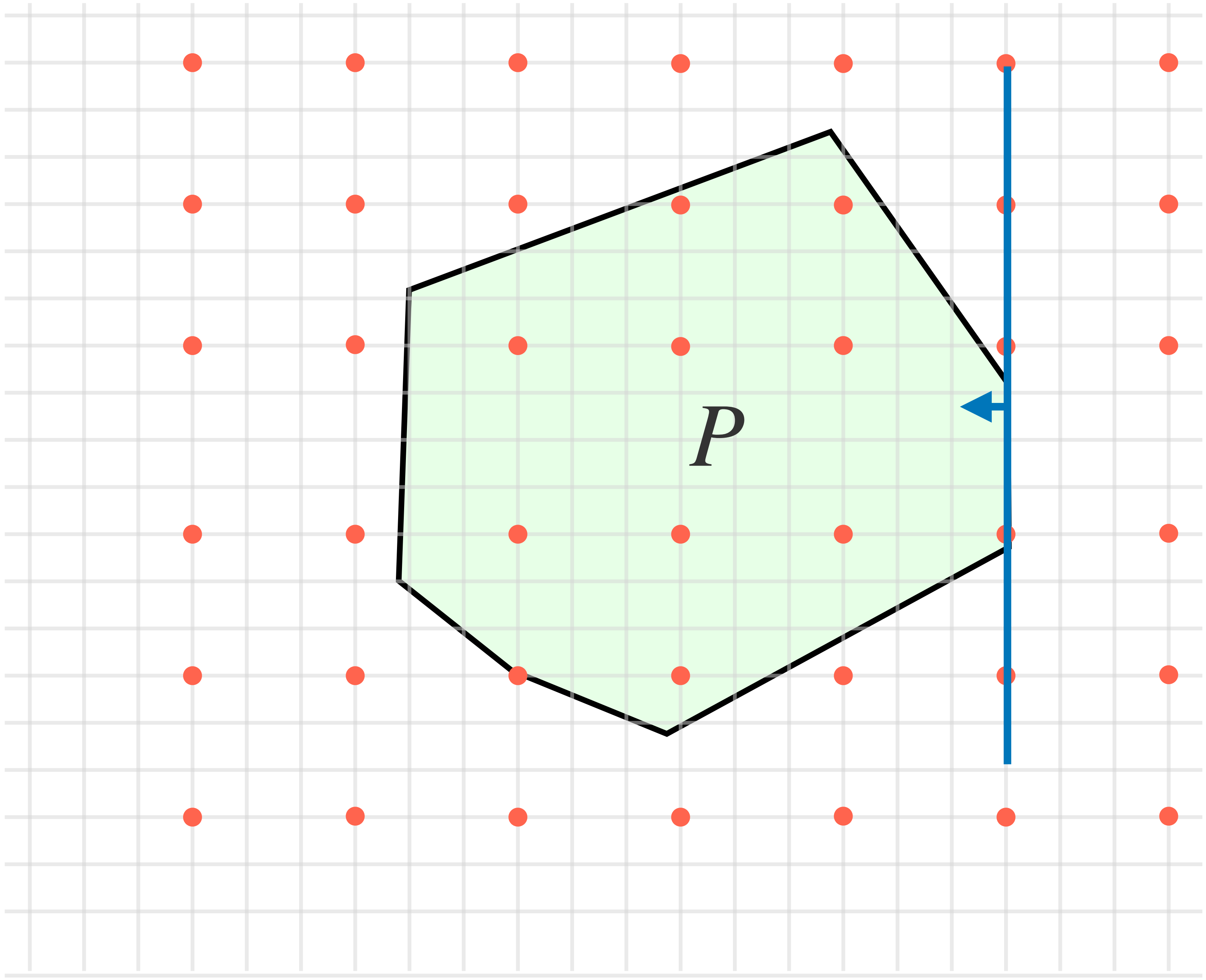

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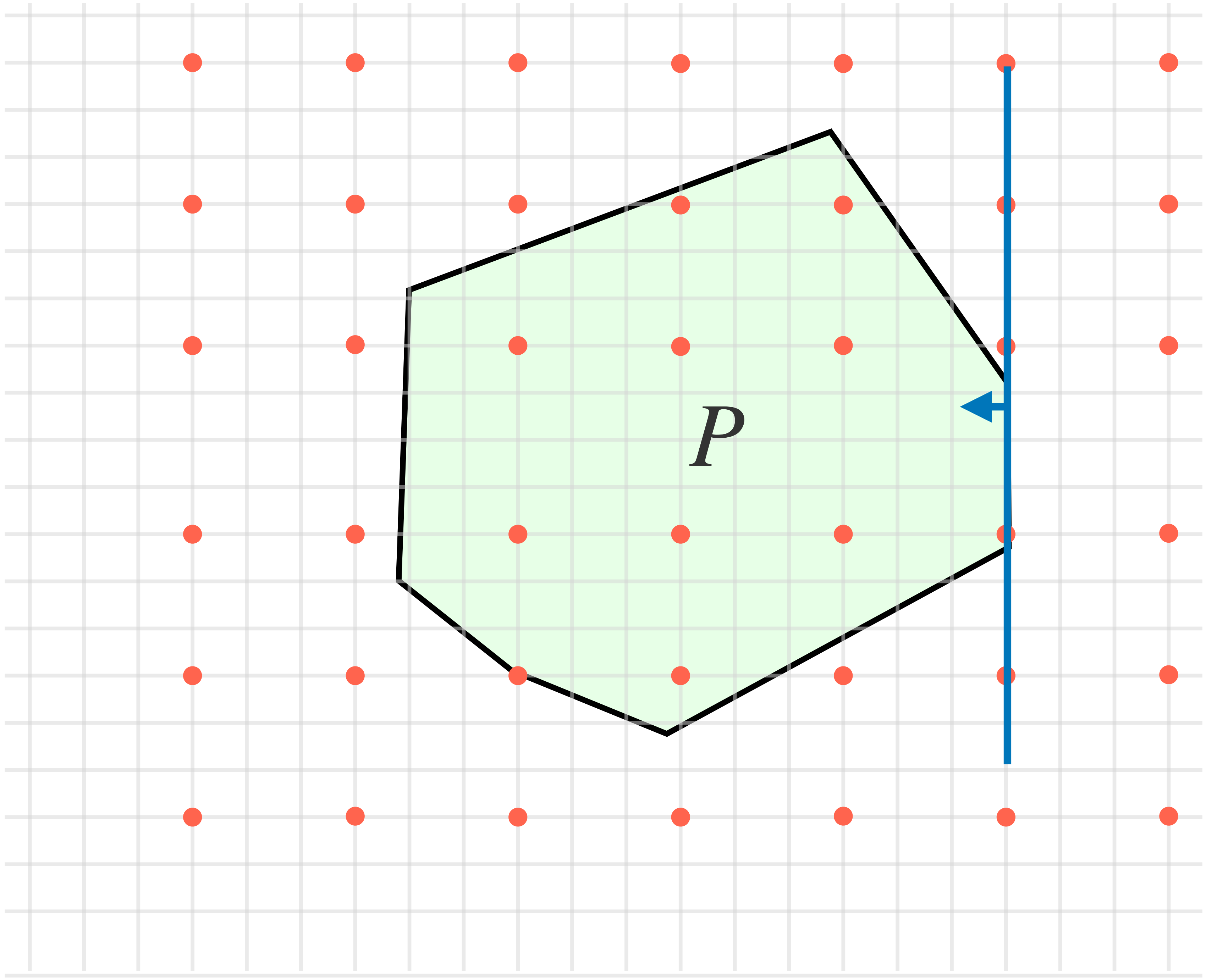

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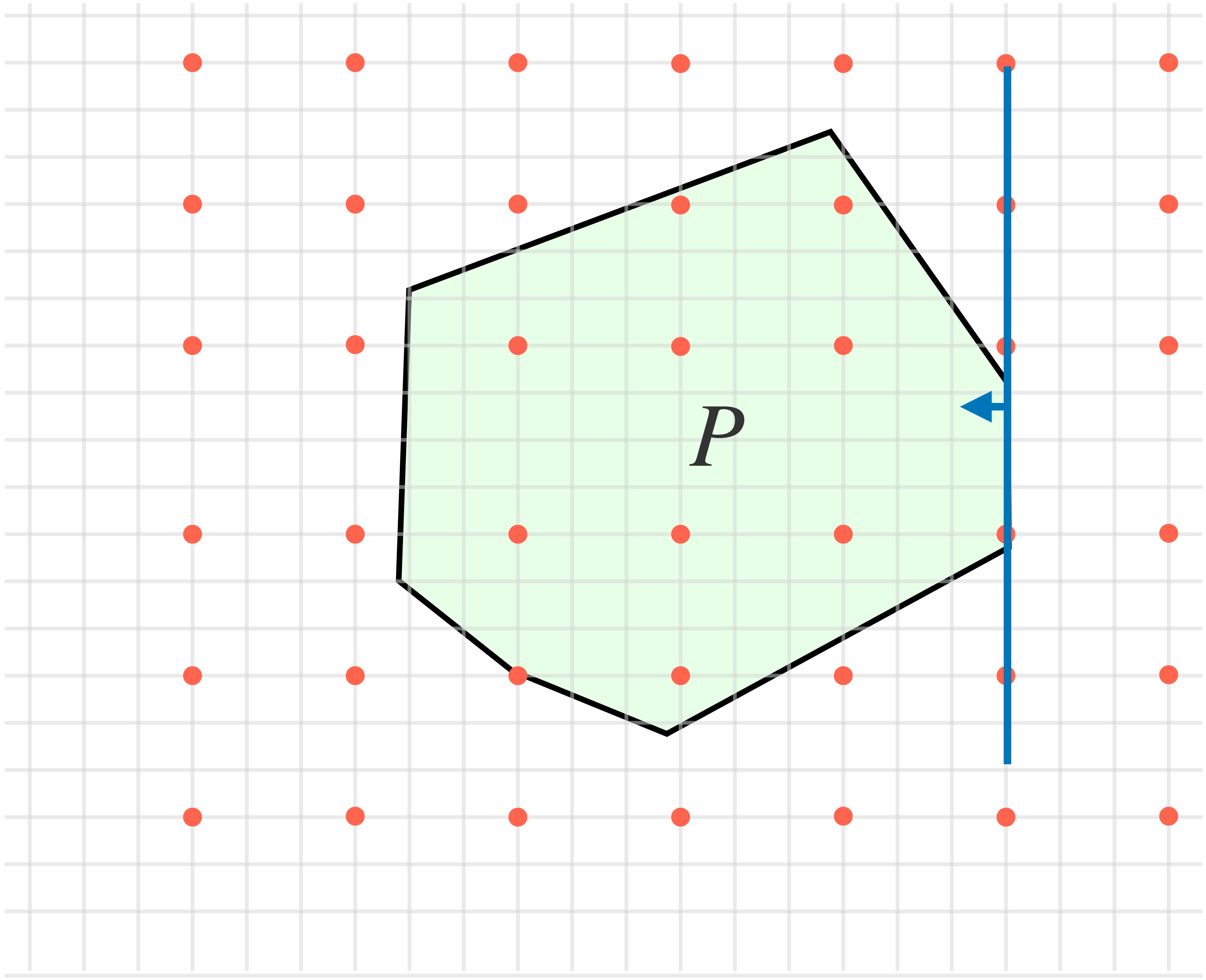

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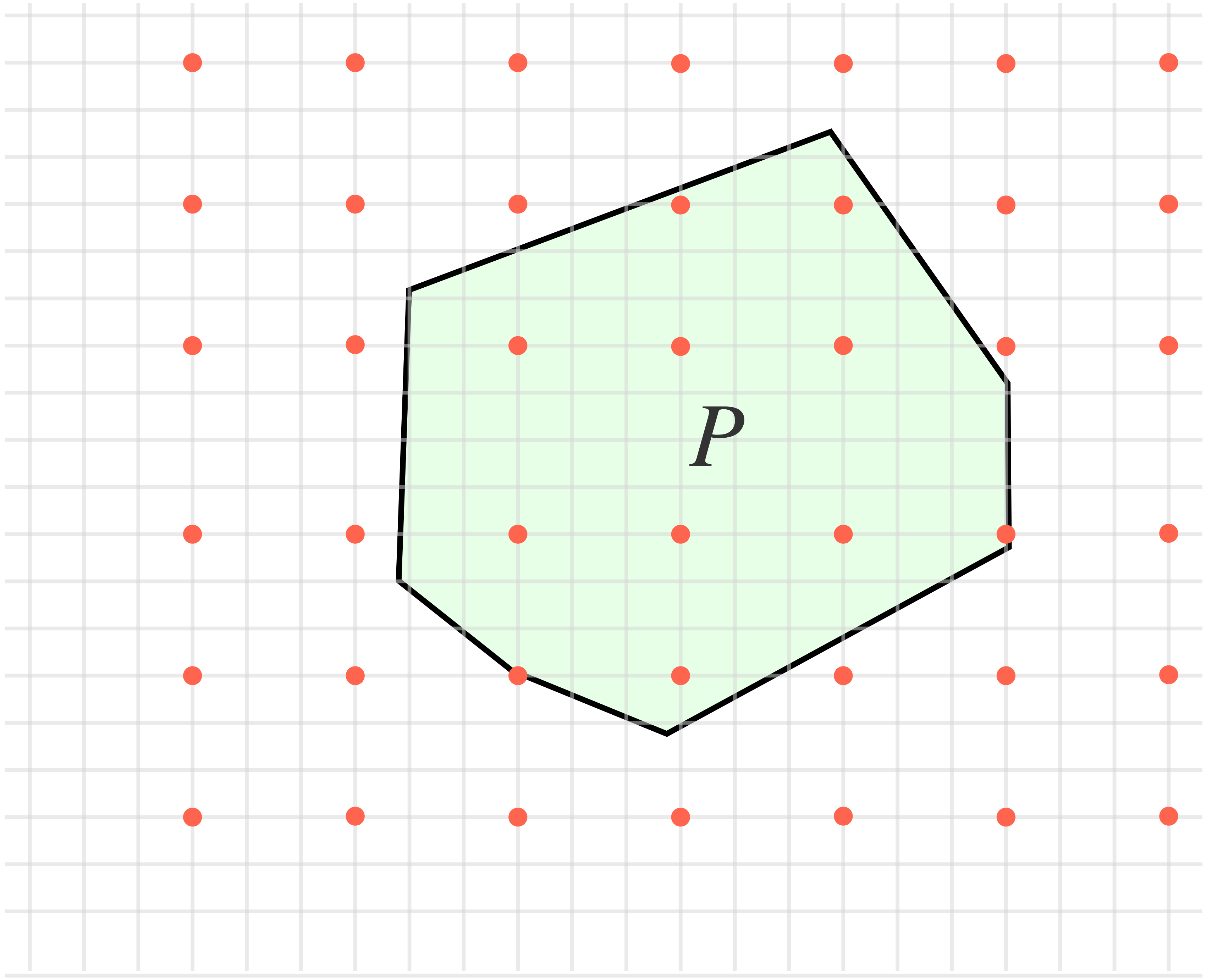
Combine cutting planes with branch-and-bound



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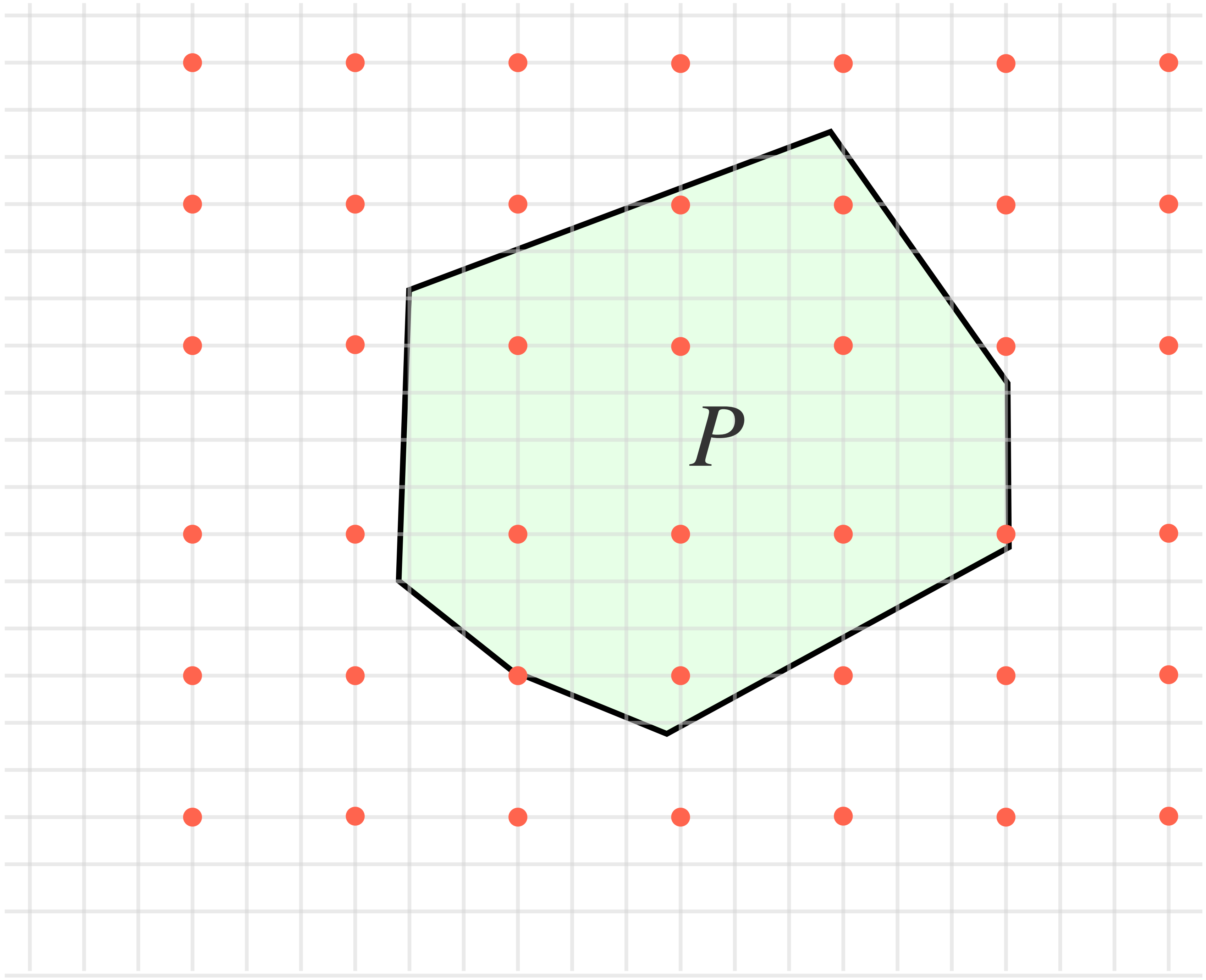

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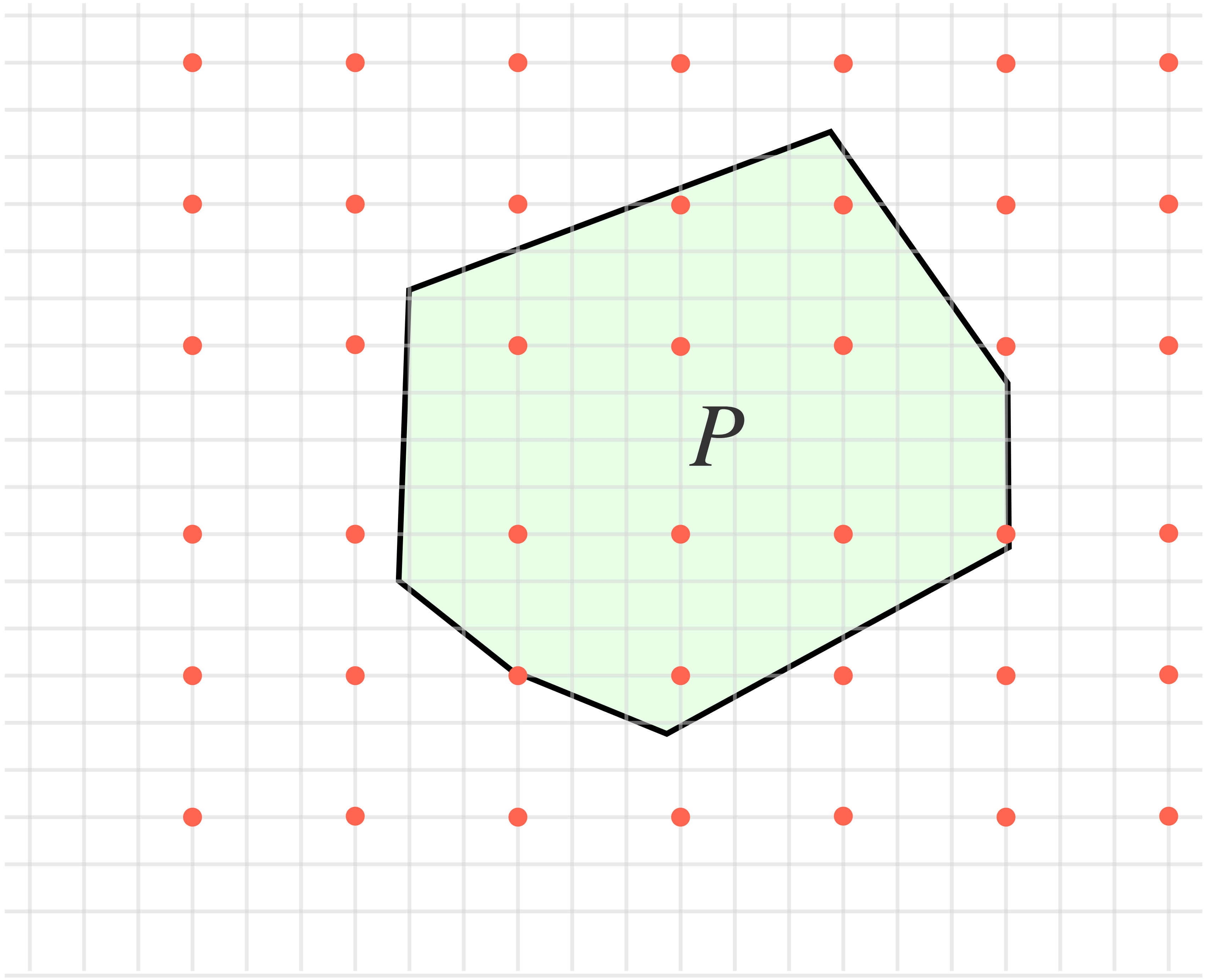

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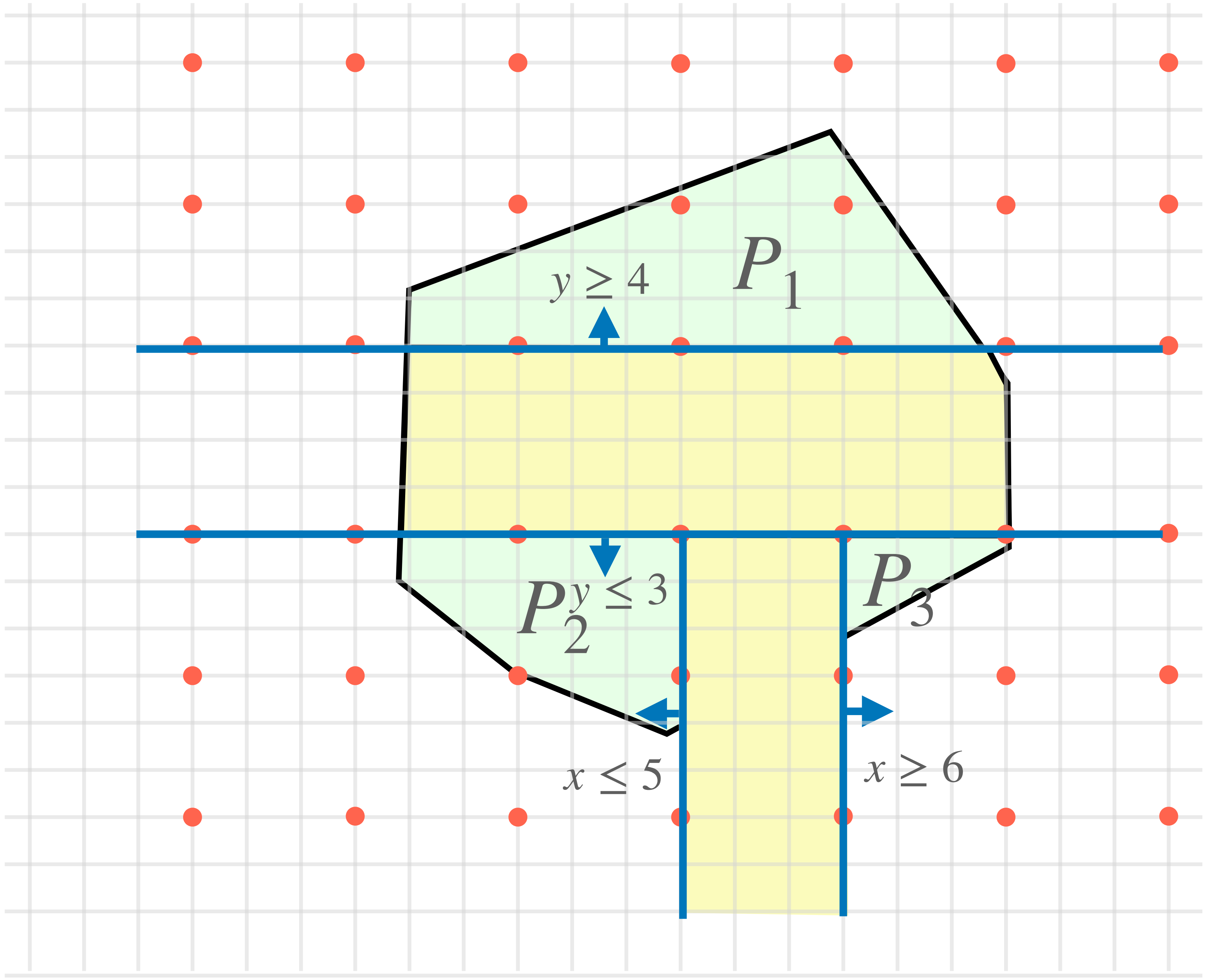

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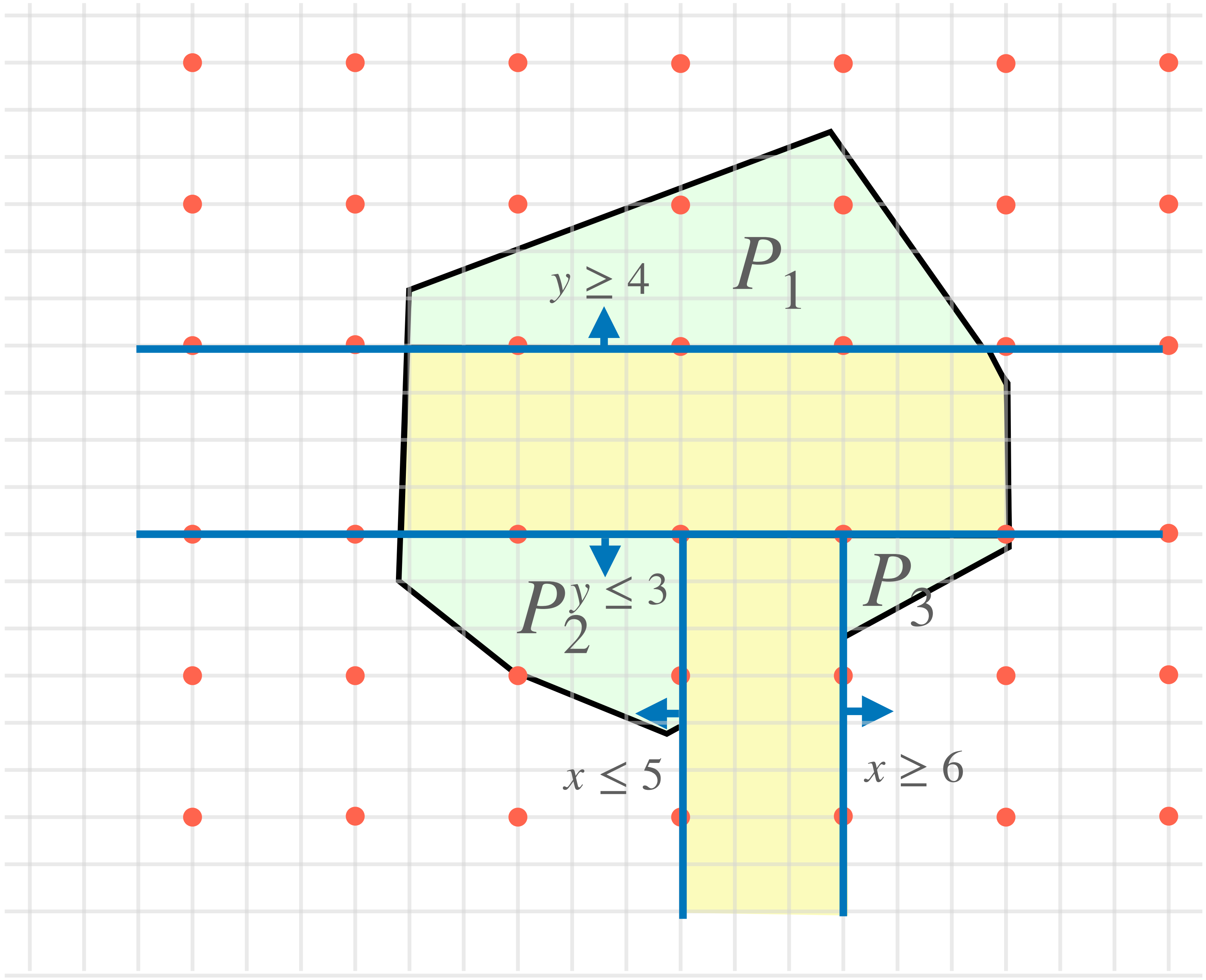

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All integer points preserved!

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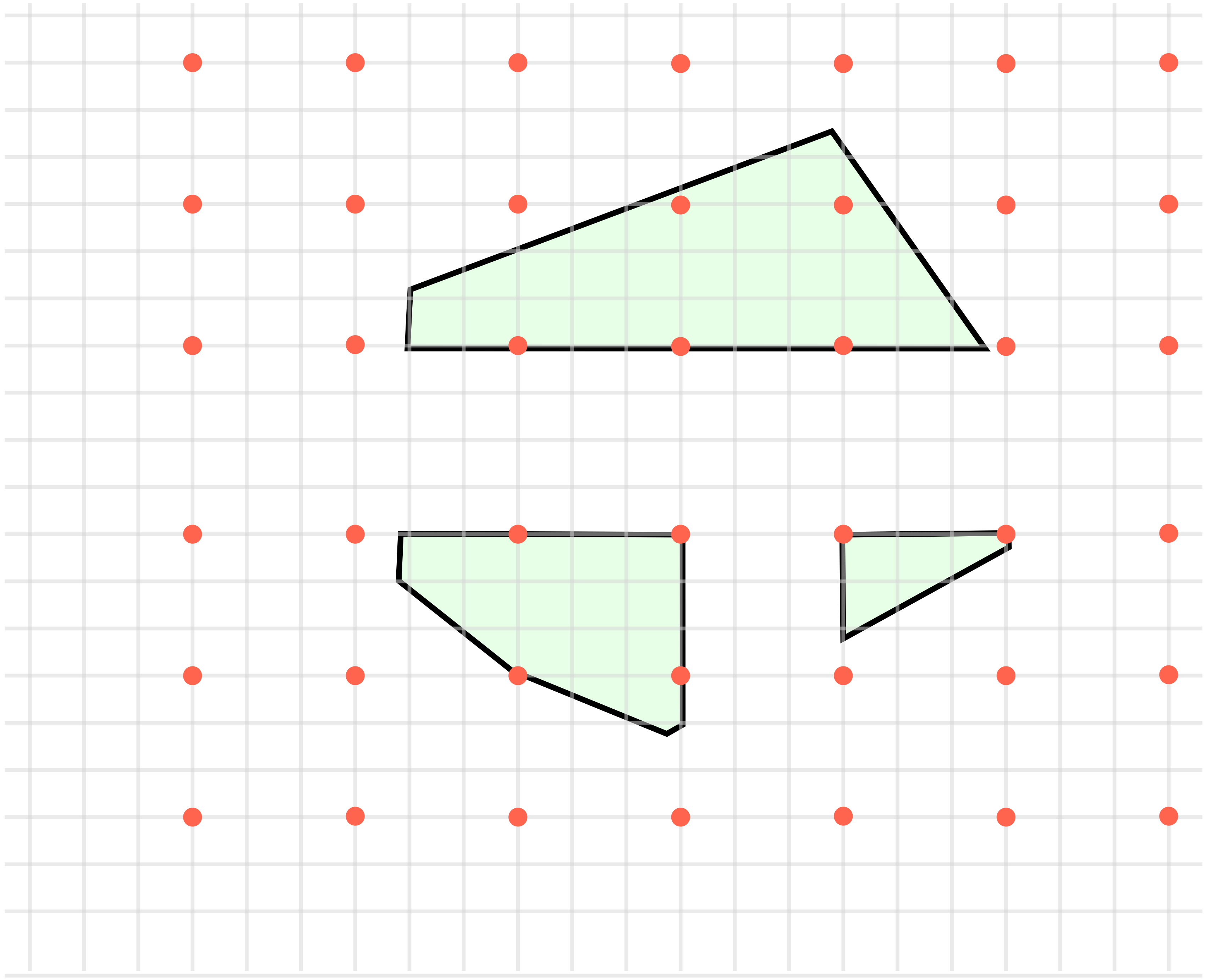

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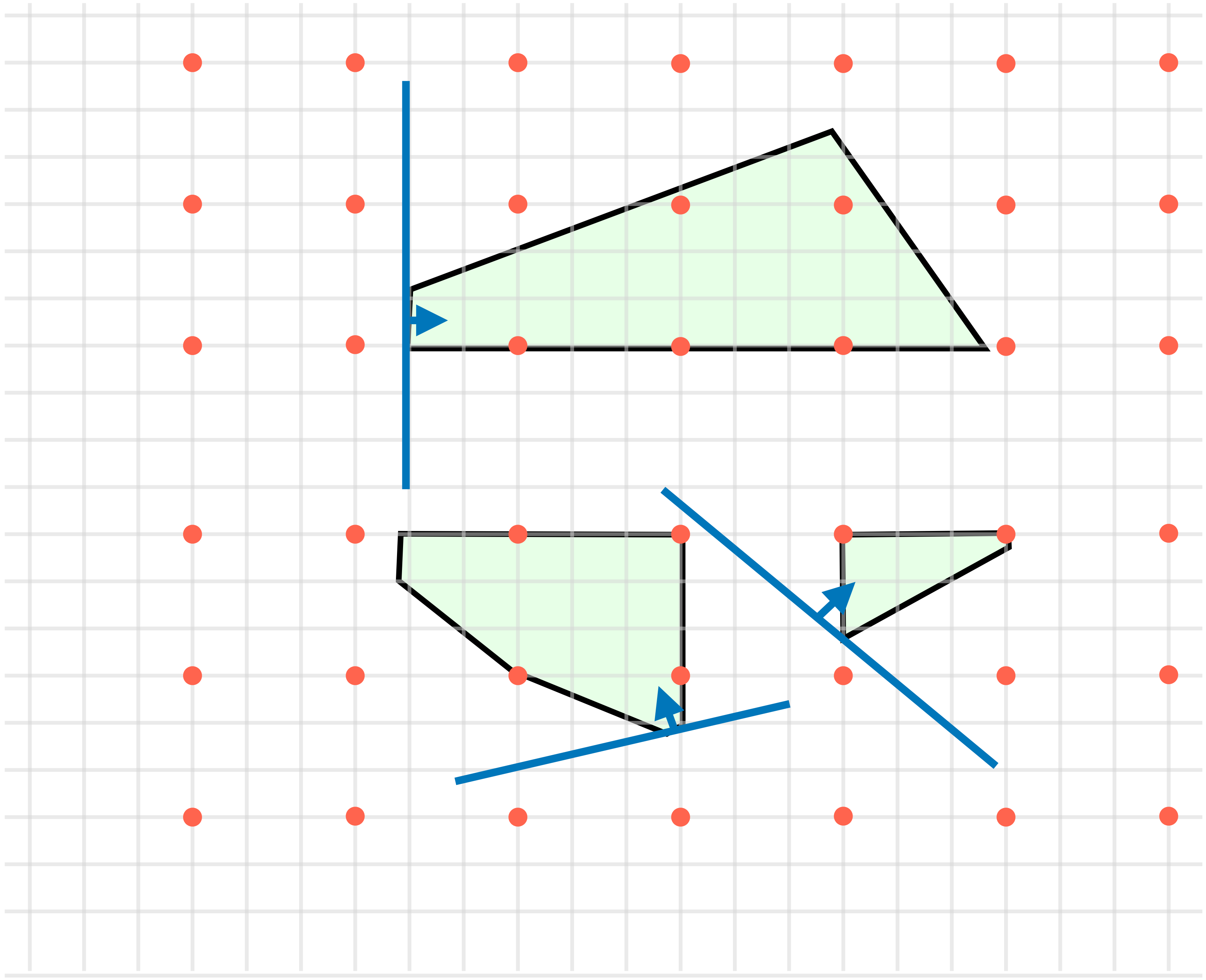

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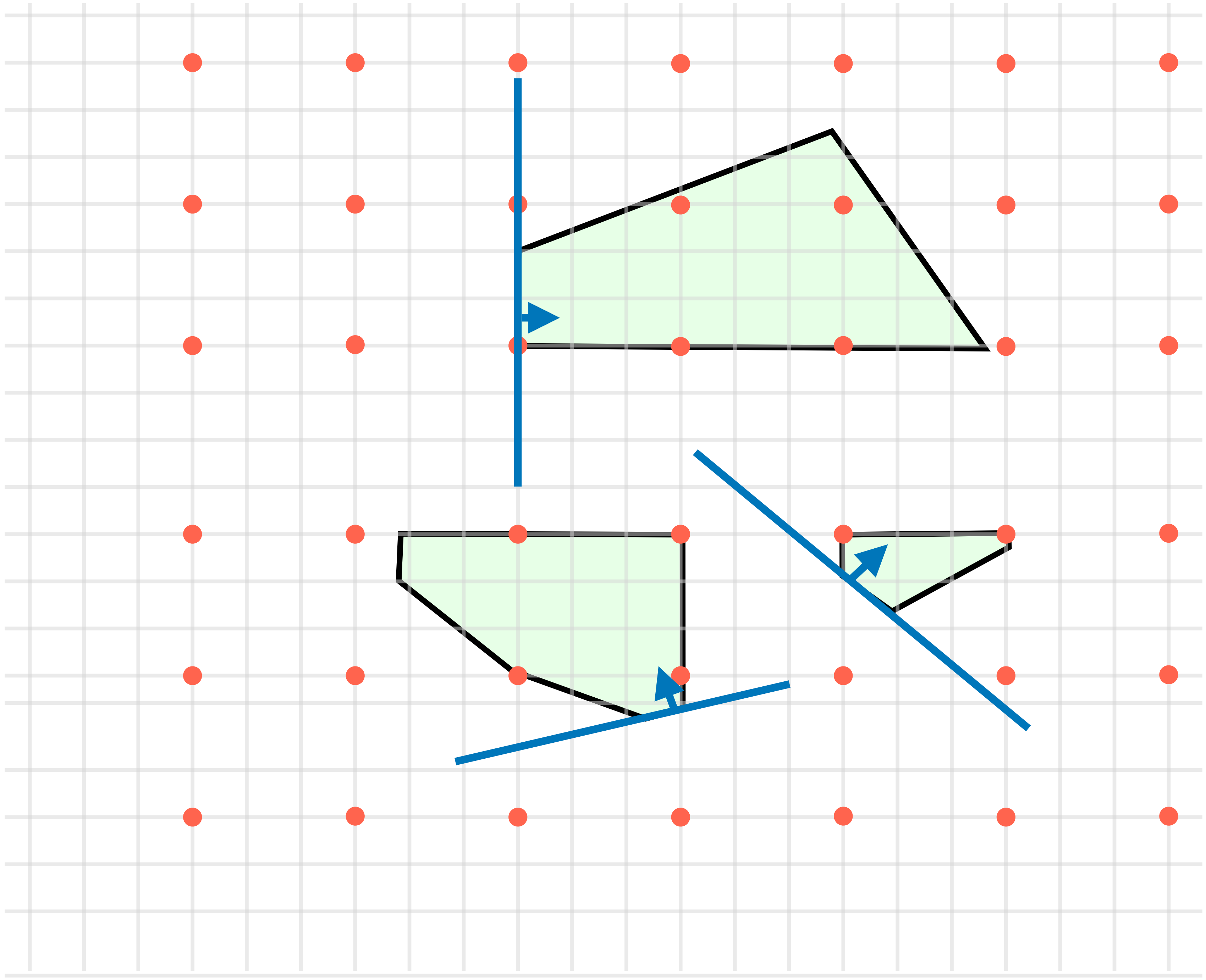

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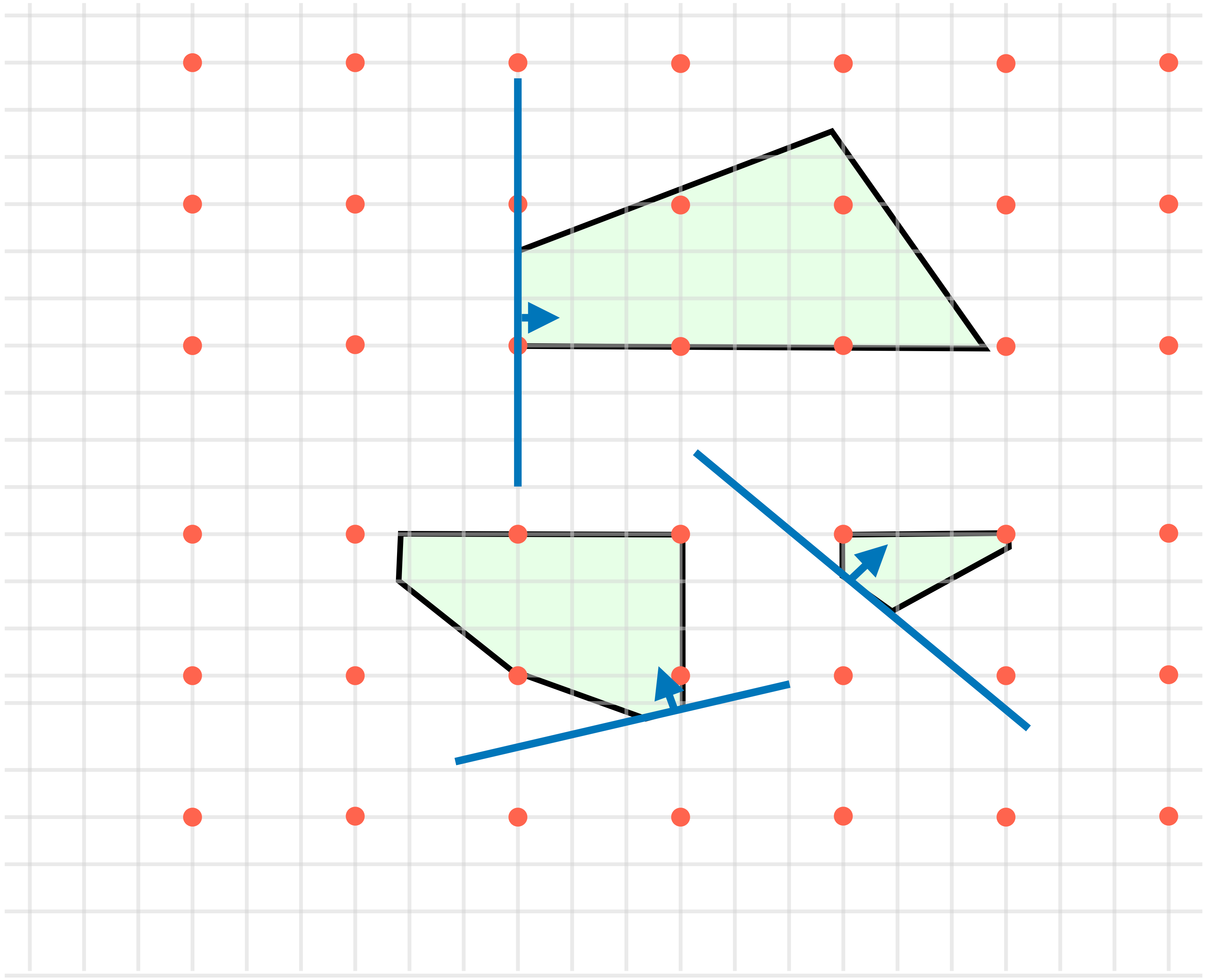

In practice branching is done by splitting on $ax \leq b$ and $ax \geq b + 1$ for $a \in \mathbb{Z}^n, b \in \mathbb{Z}$

Analyzing Modern IP Algorithms

Modern IP algorithms are a complicated mess of heuristics:

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Makes analyzing these algorithms directly challenging!

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Instead of trying to understand an algorithm A directly, formalize the techniques used by the algorithms into a proof system S.

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Even works in optimization! Algorithm has to “prove” that no better solution exists

Instead of trying to understand an algorithm A directly, formalize the techniques used by the algorithms into a proof system S.

$\rightarrow$ Lower bounds on the size of S-proofs imply runtime lower bounds for A

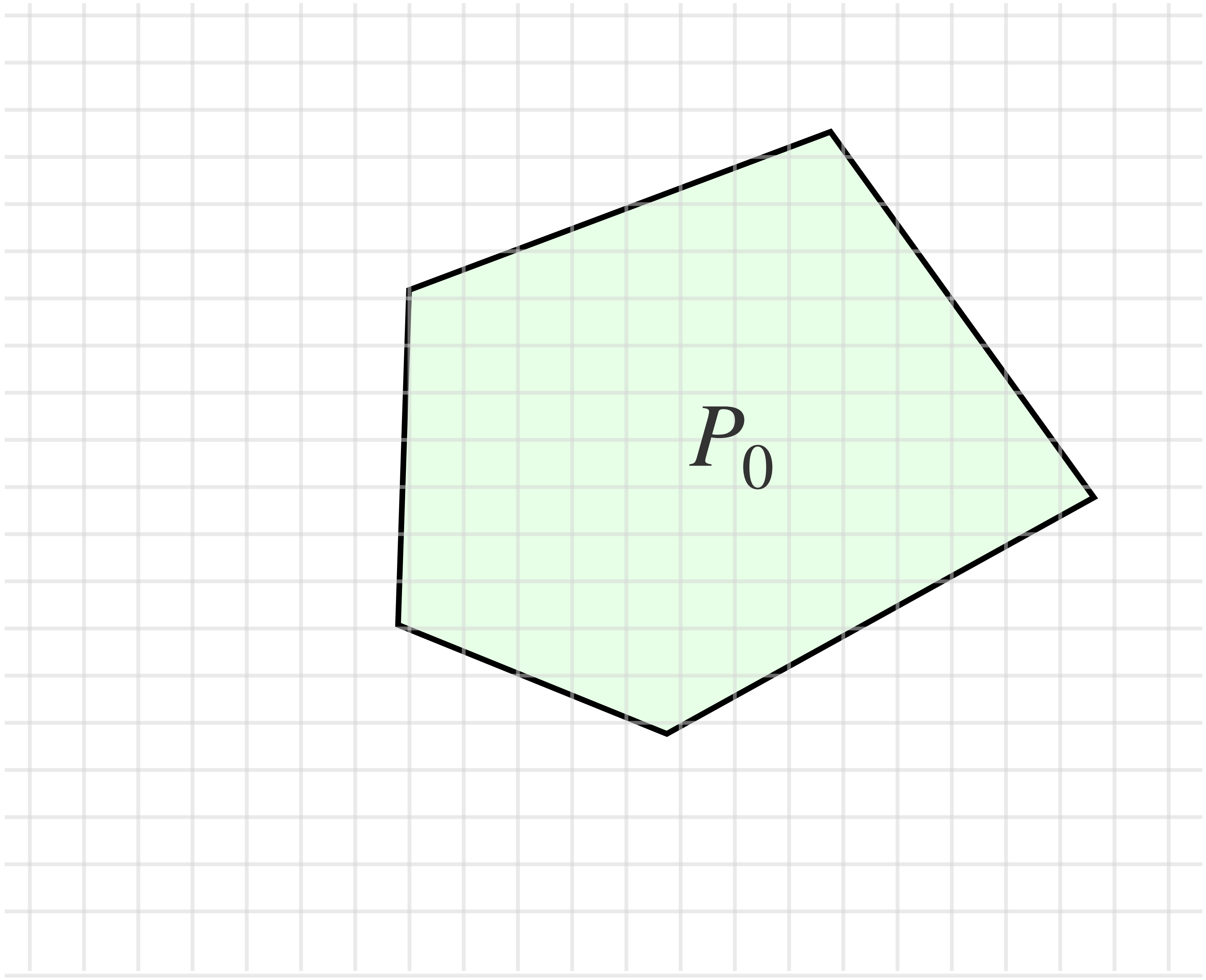
Cutting Planes

A Cutting Planes refutation of P with $P \cap \mathbb{Z}^n = \emptyset$ is a sequence of polytopes

$$(P =)\ P_0, \ldots, P_s\ (= \emptyset)$$

where P_i is obtained by a CG-cut from P_{i-1}.

The size of the proof is s.



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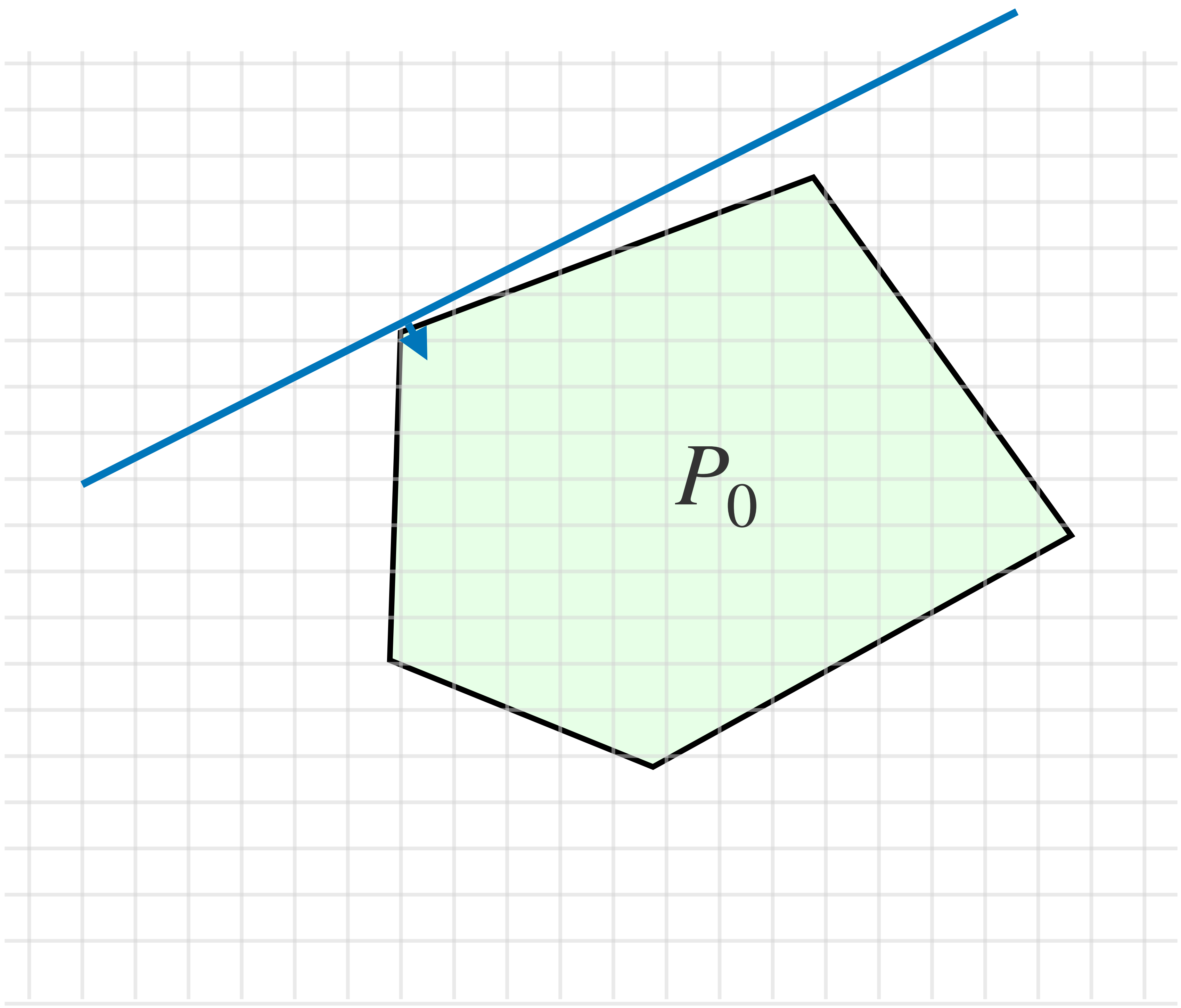

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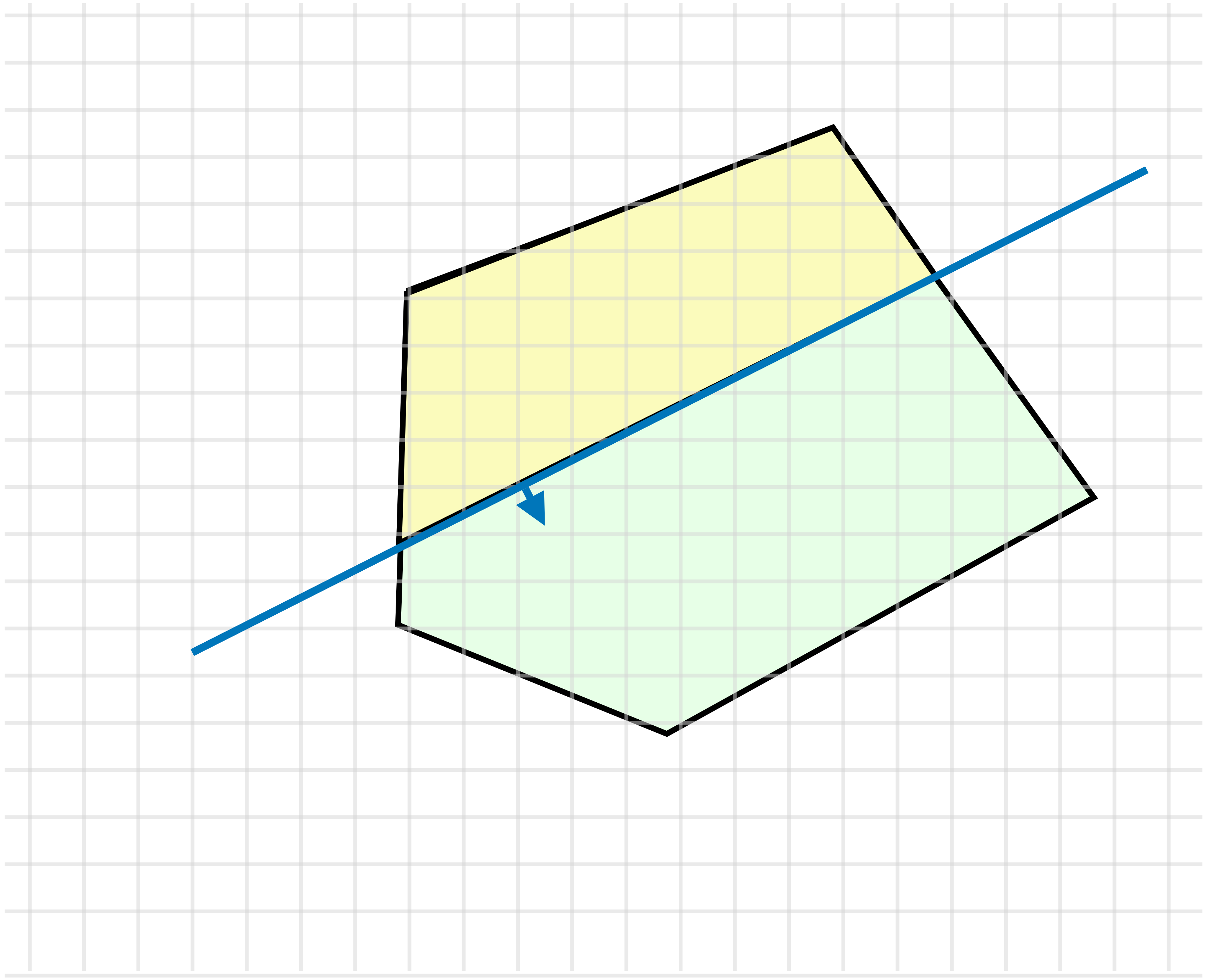

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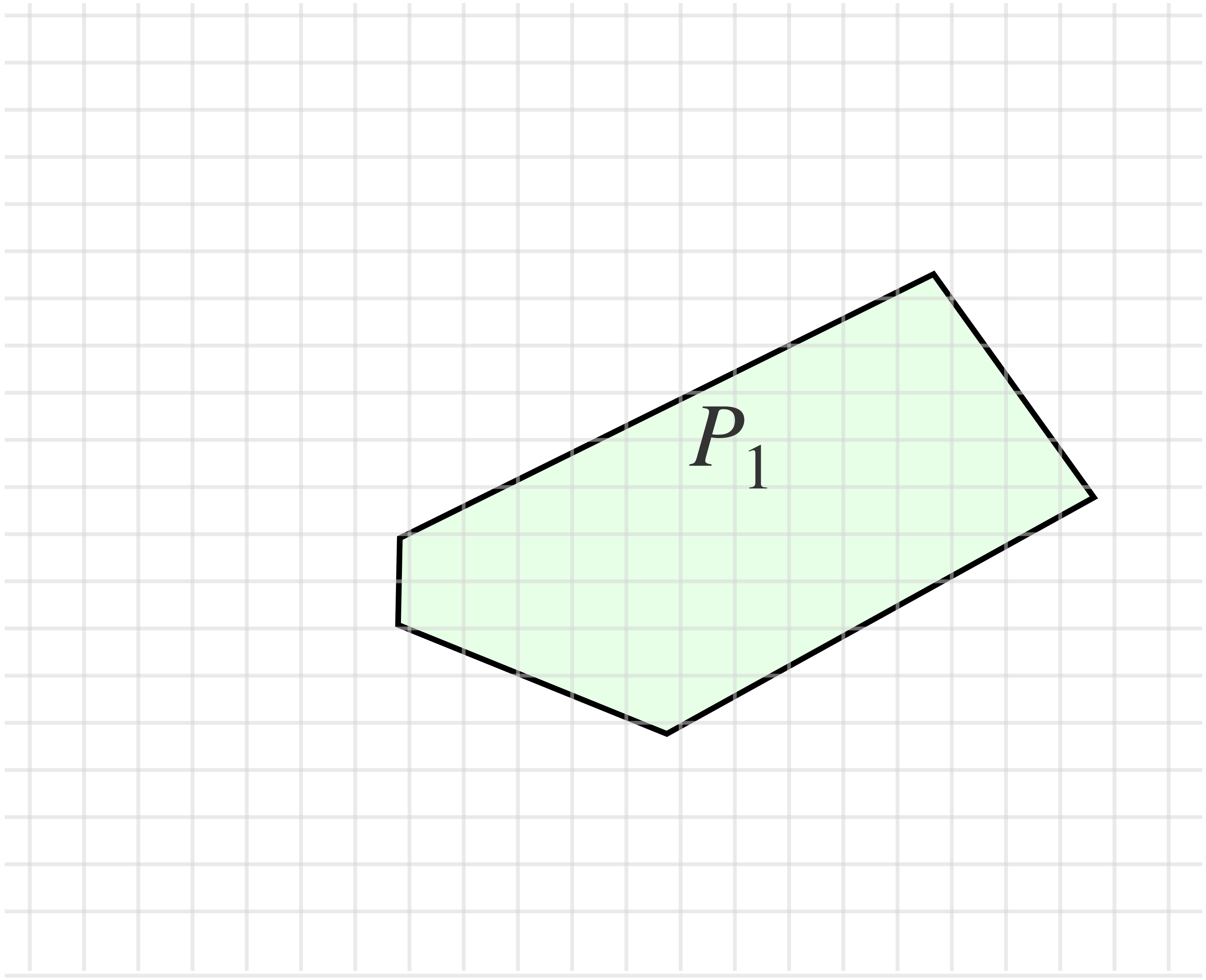

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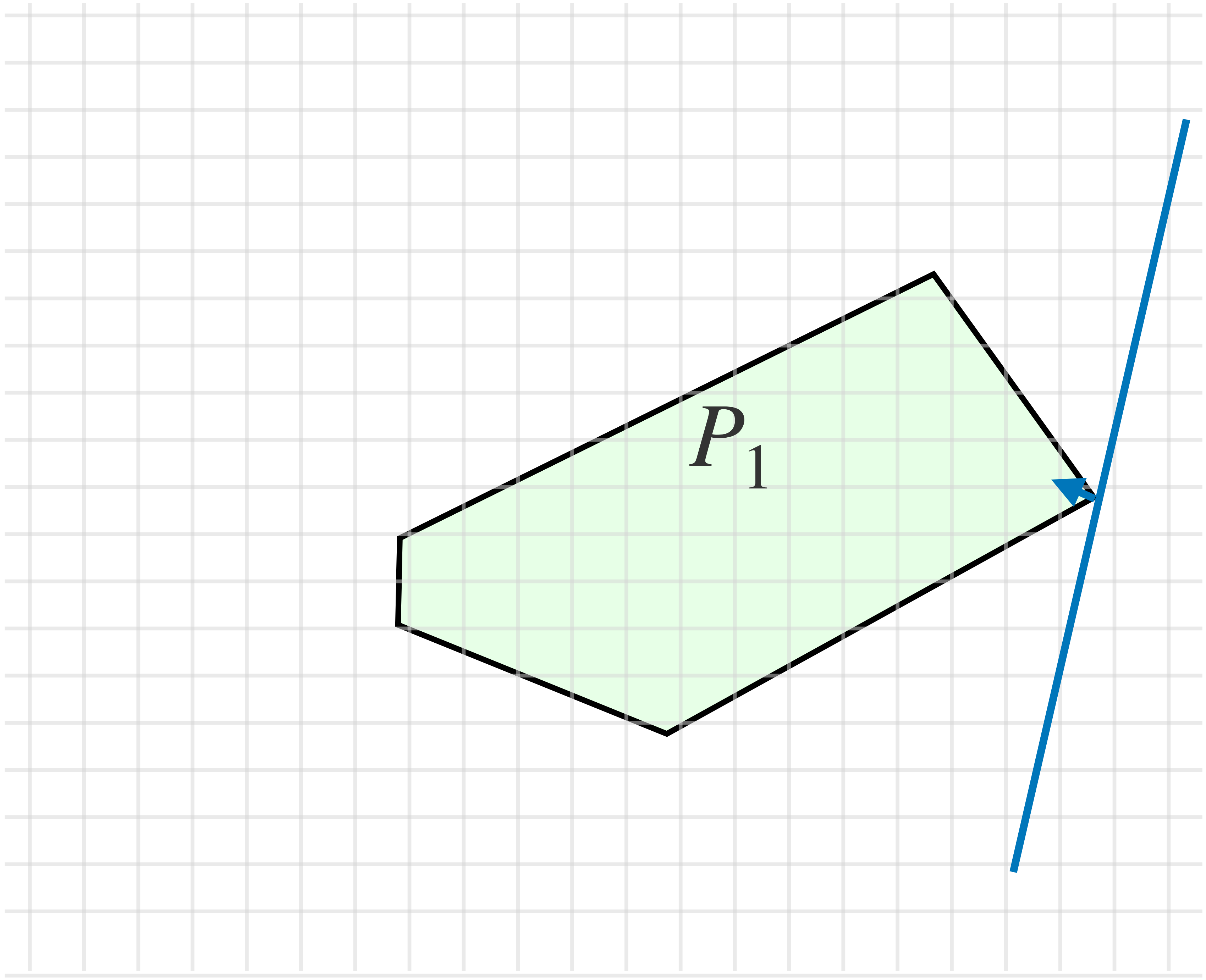

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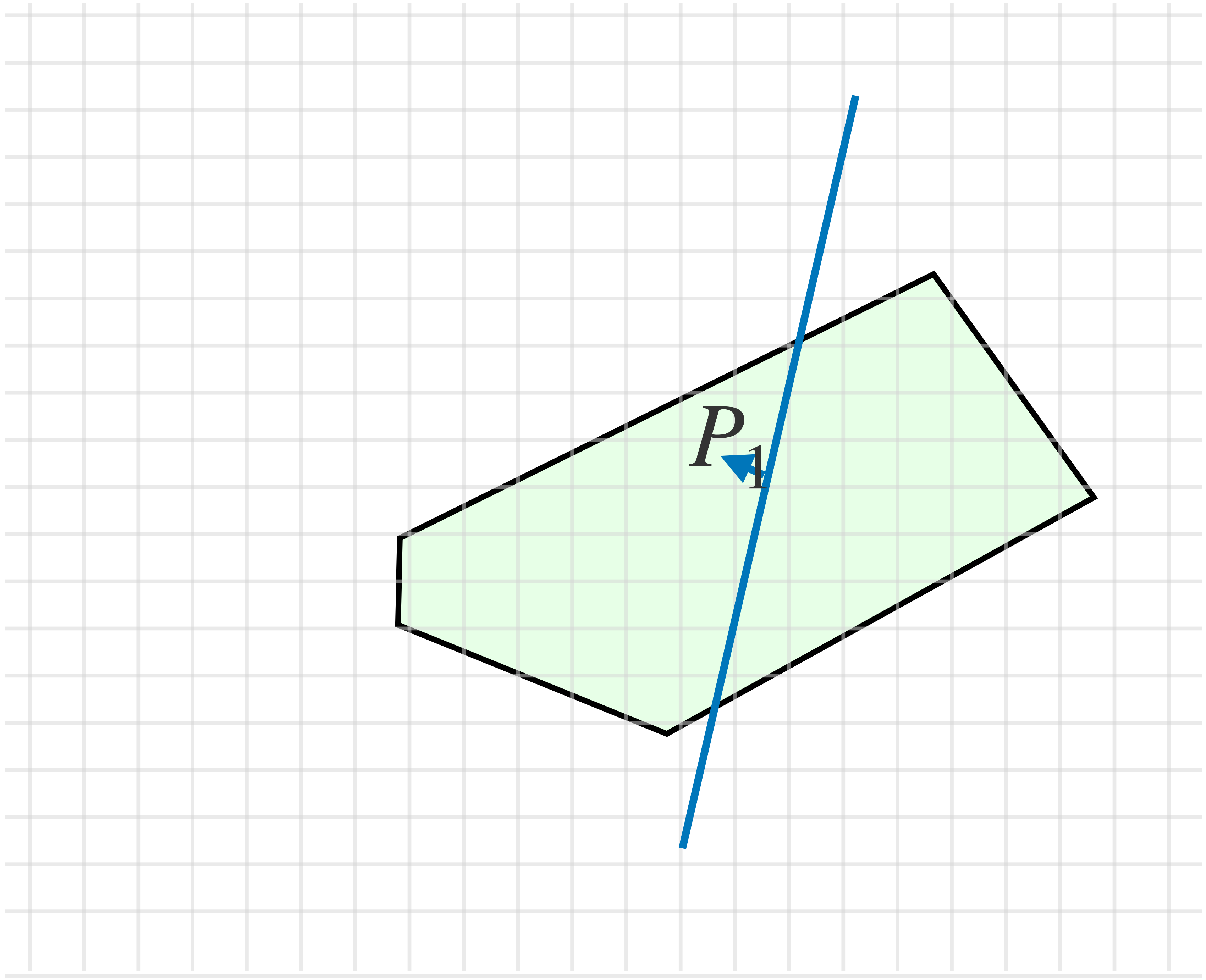

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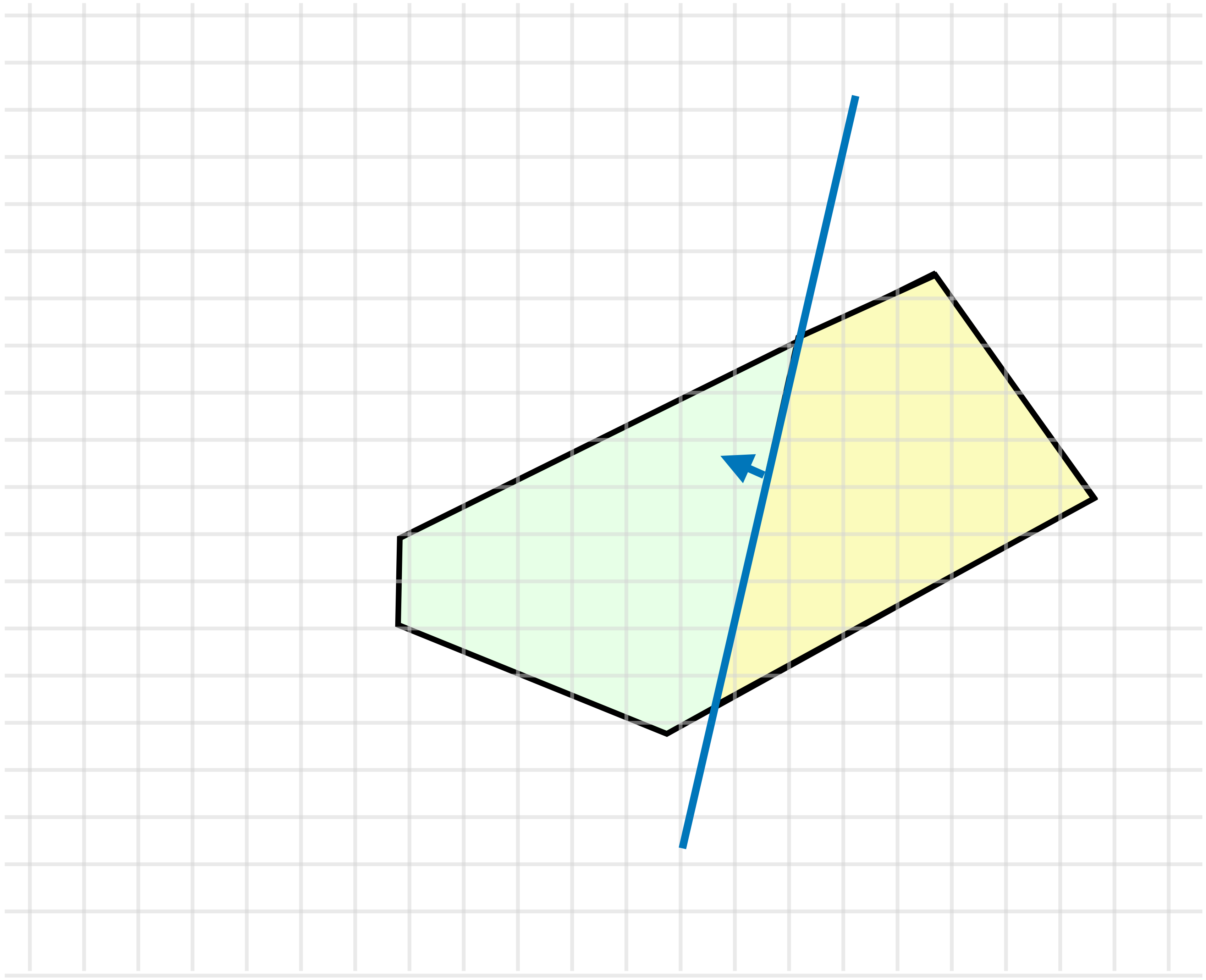

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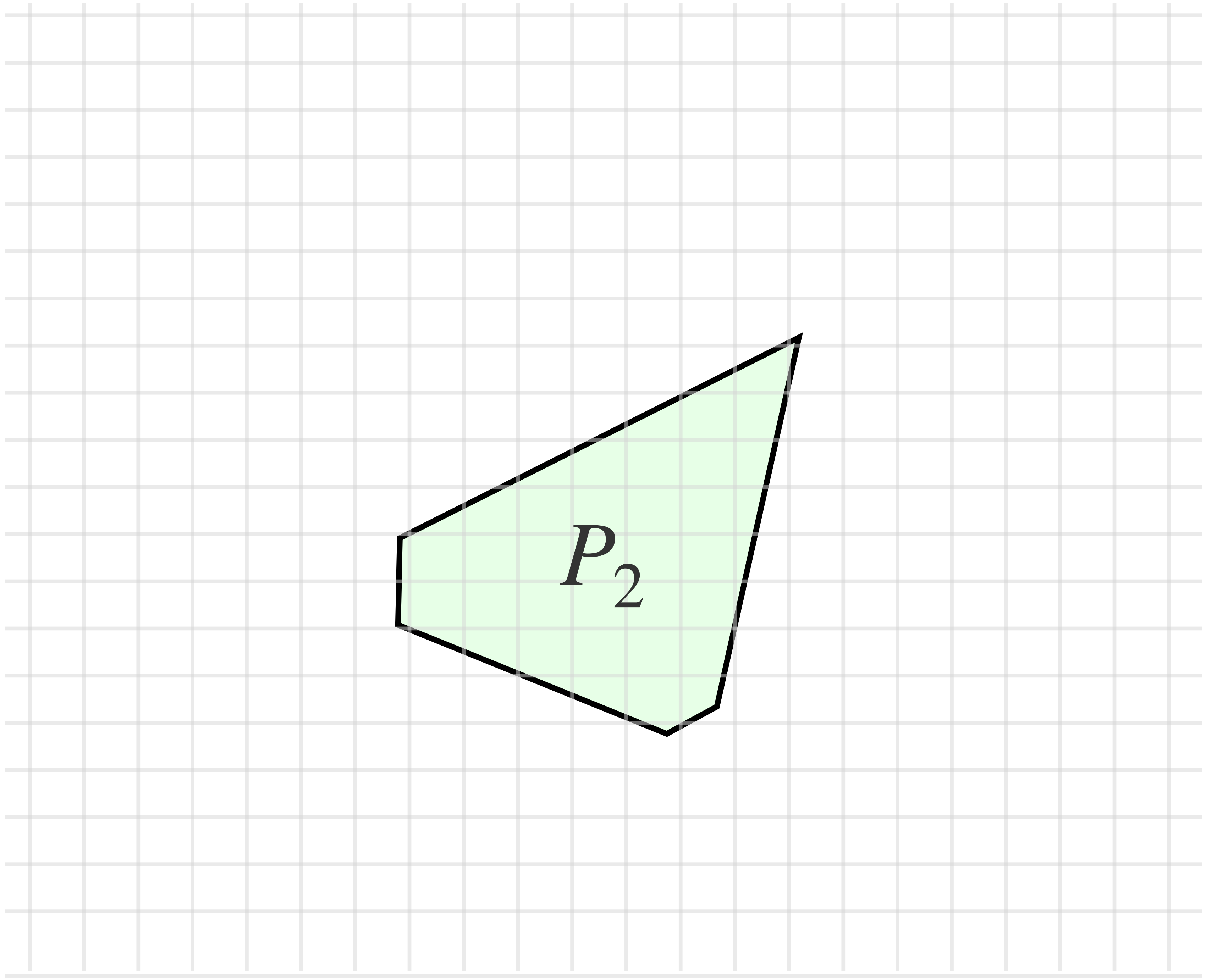

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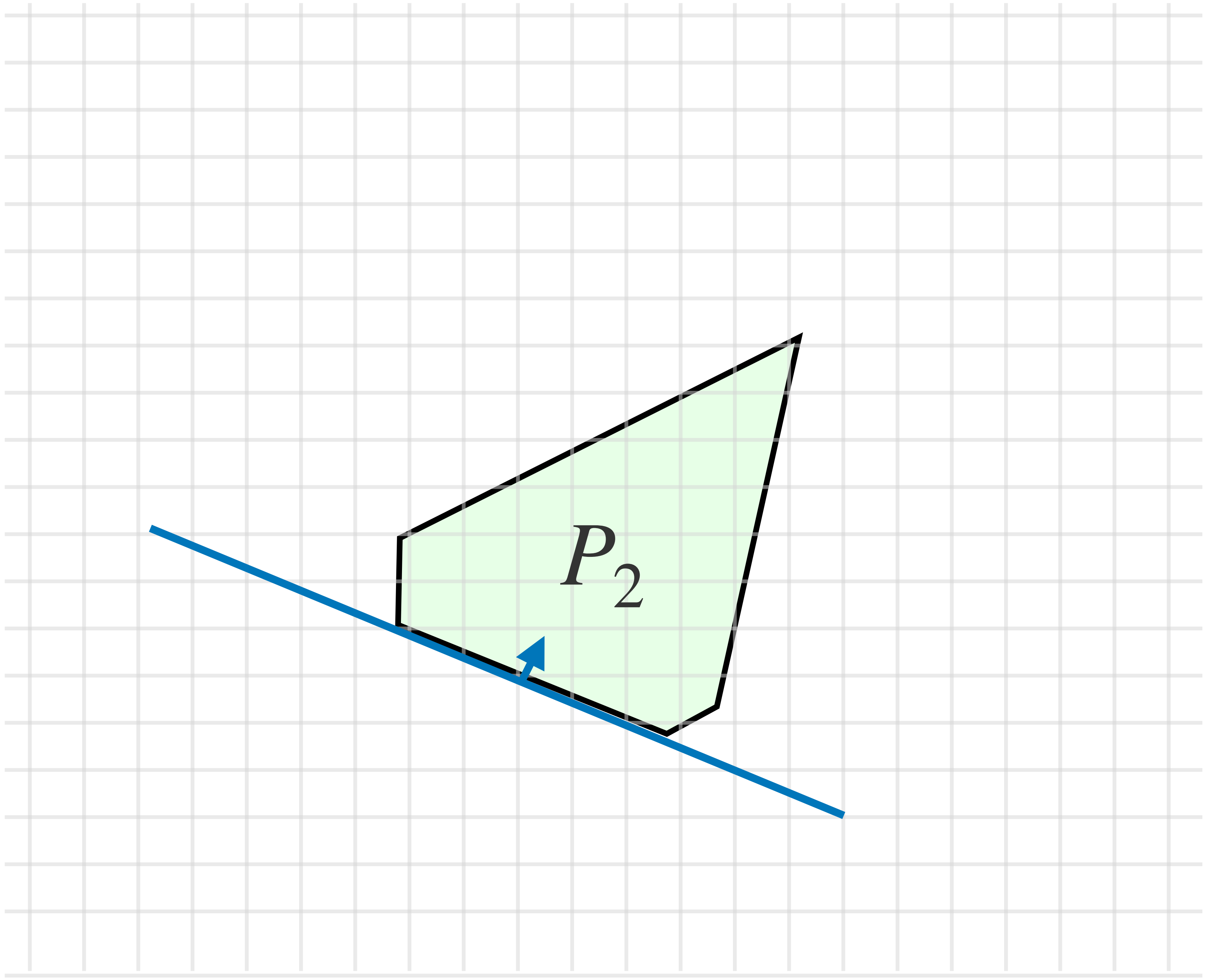

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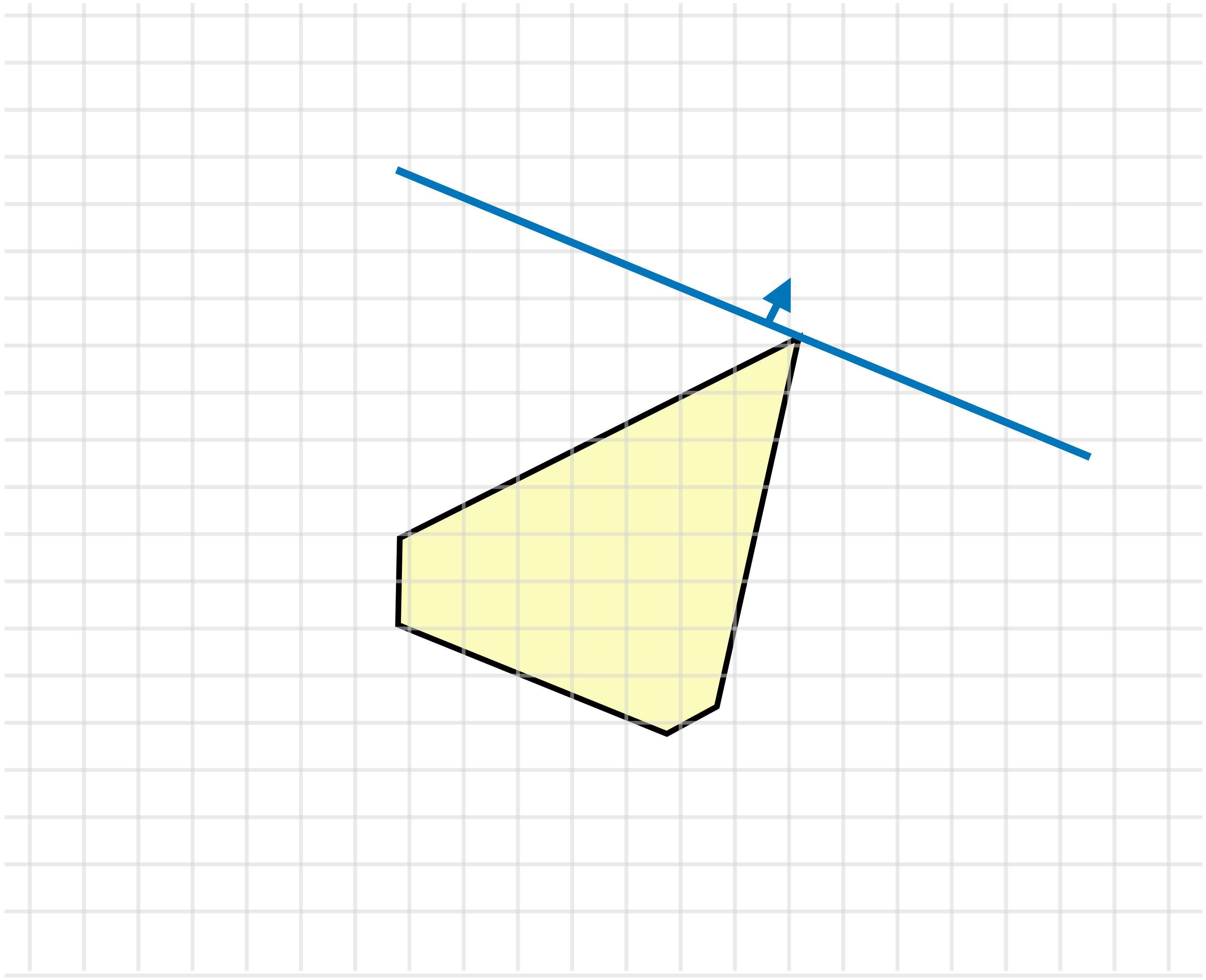

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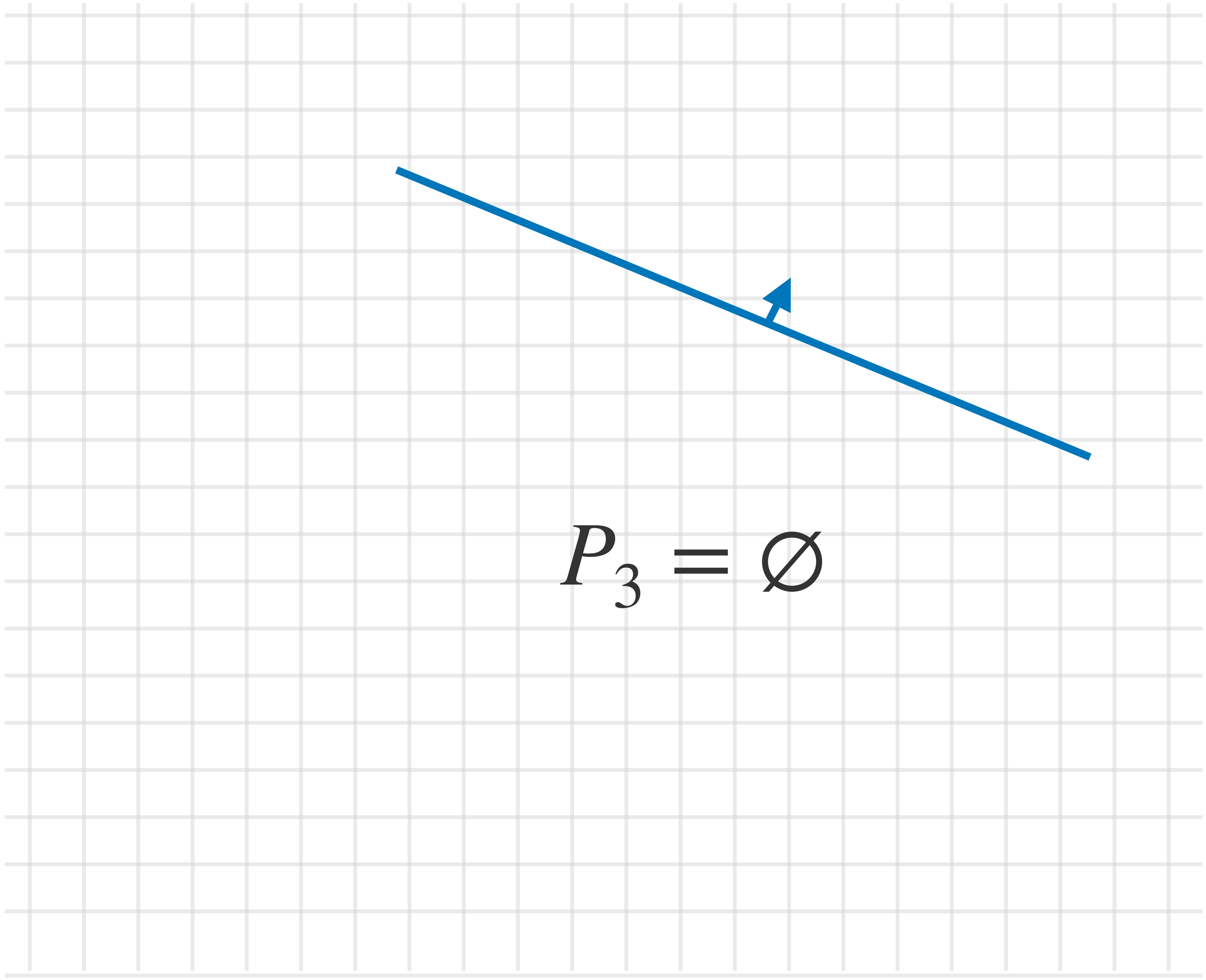

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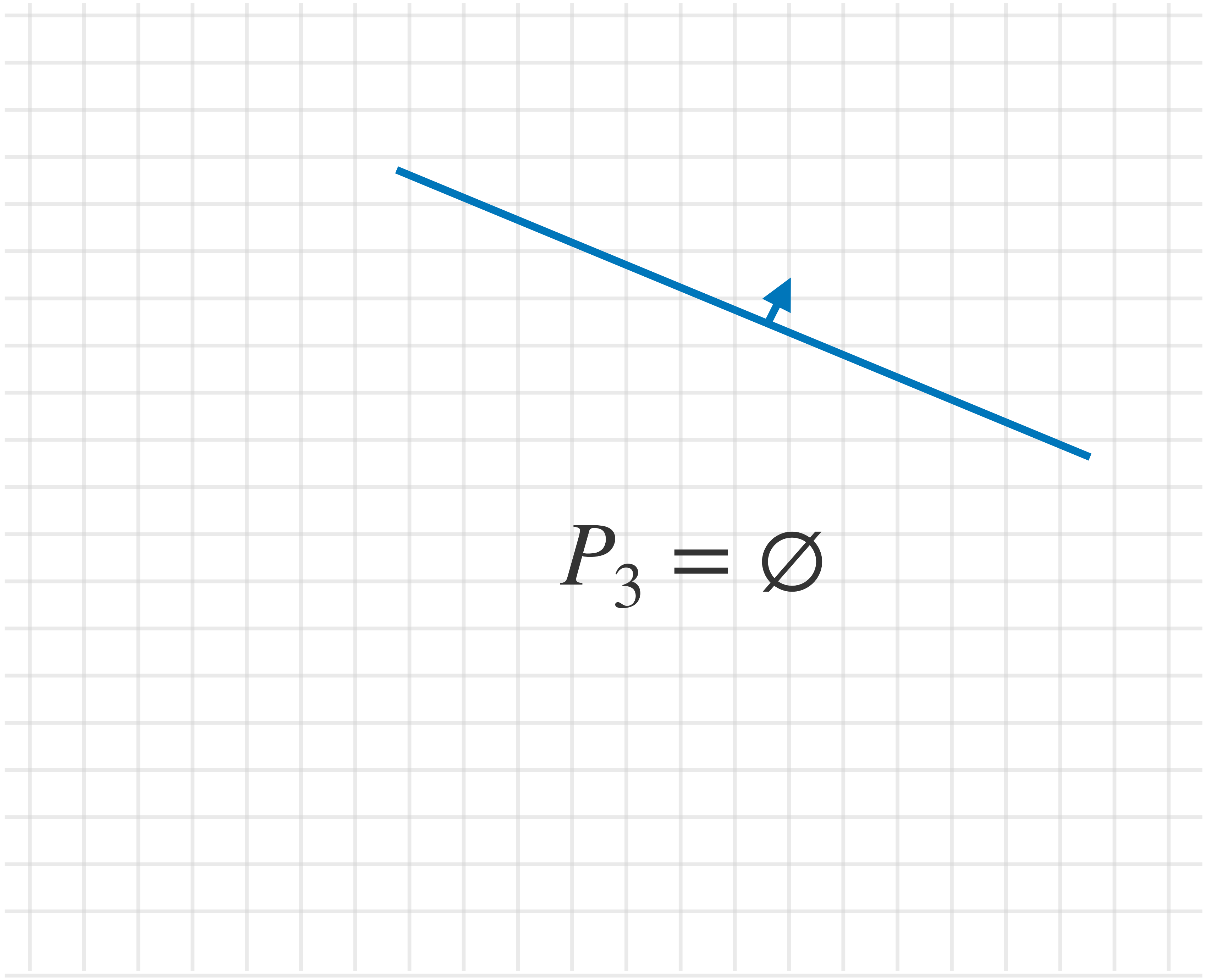

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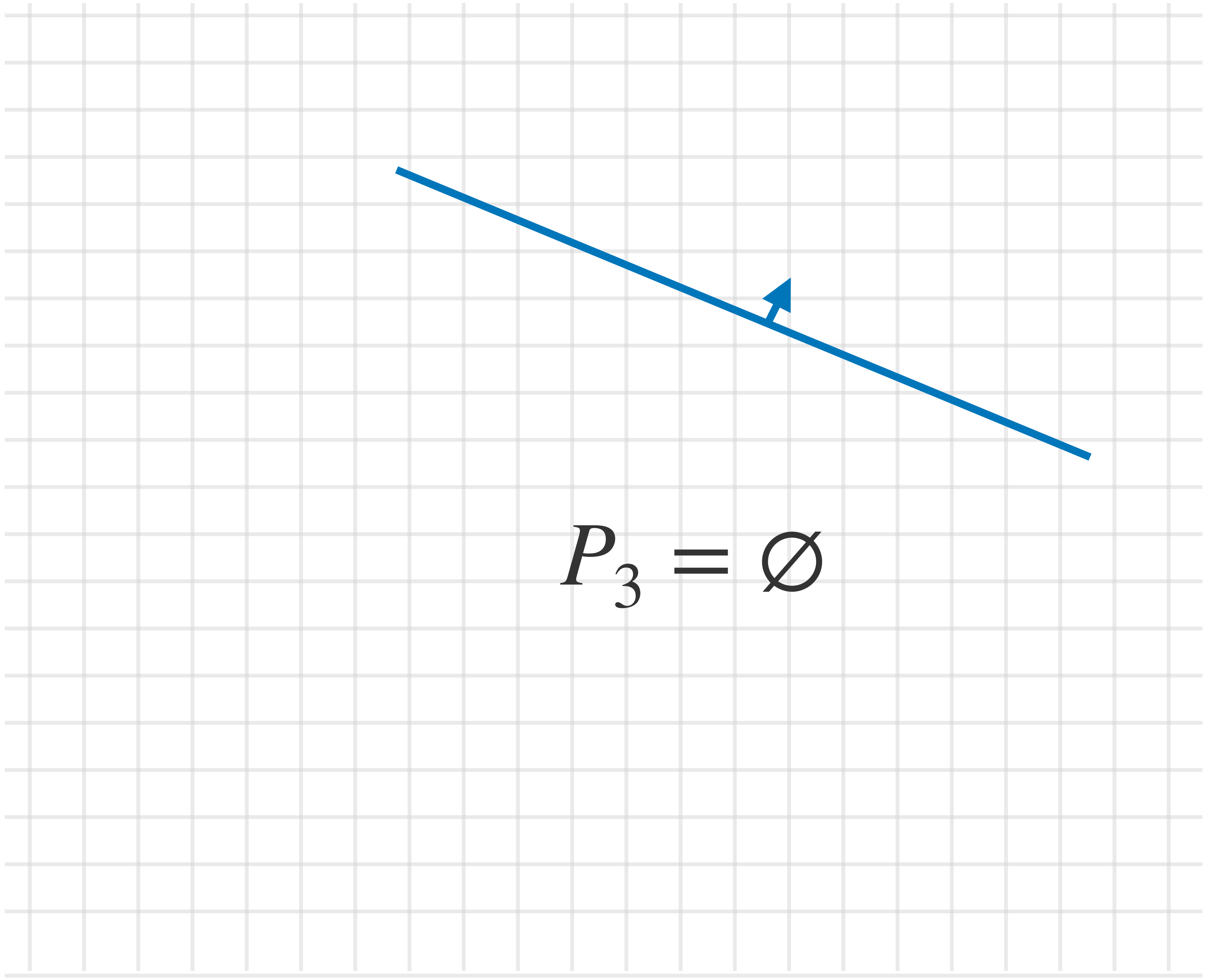

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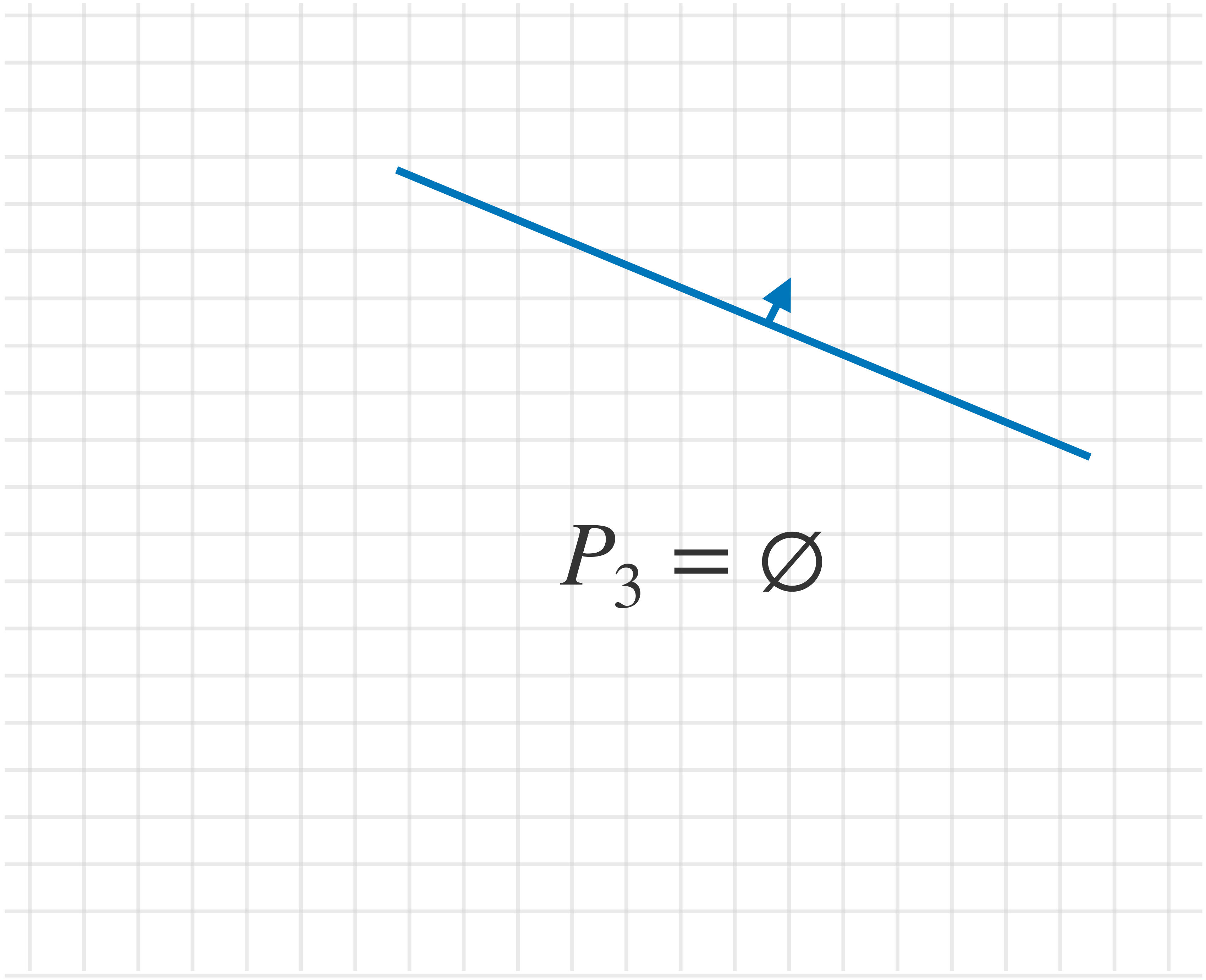

- Introduced in [Chvátal73].
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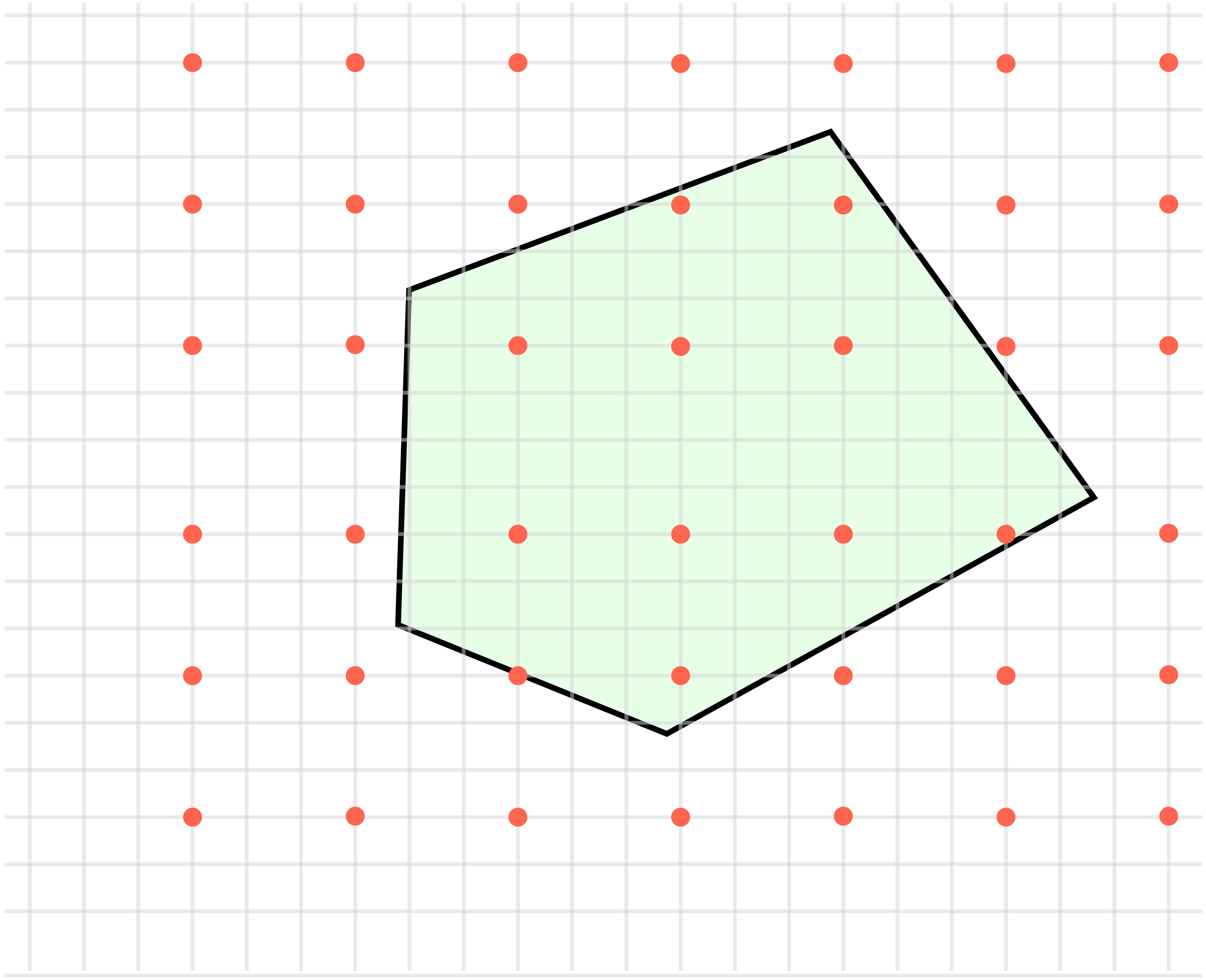
- Captures IP algorithms which use only CG-cuts (no branching).

Stabbing Planes

- Introduced to model branch-and-cut algorithms [BFI+18].



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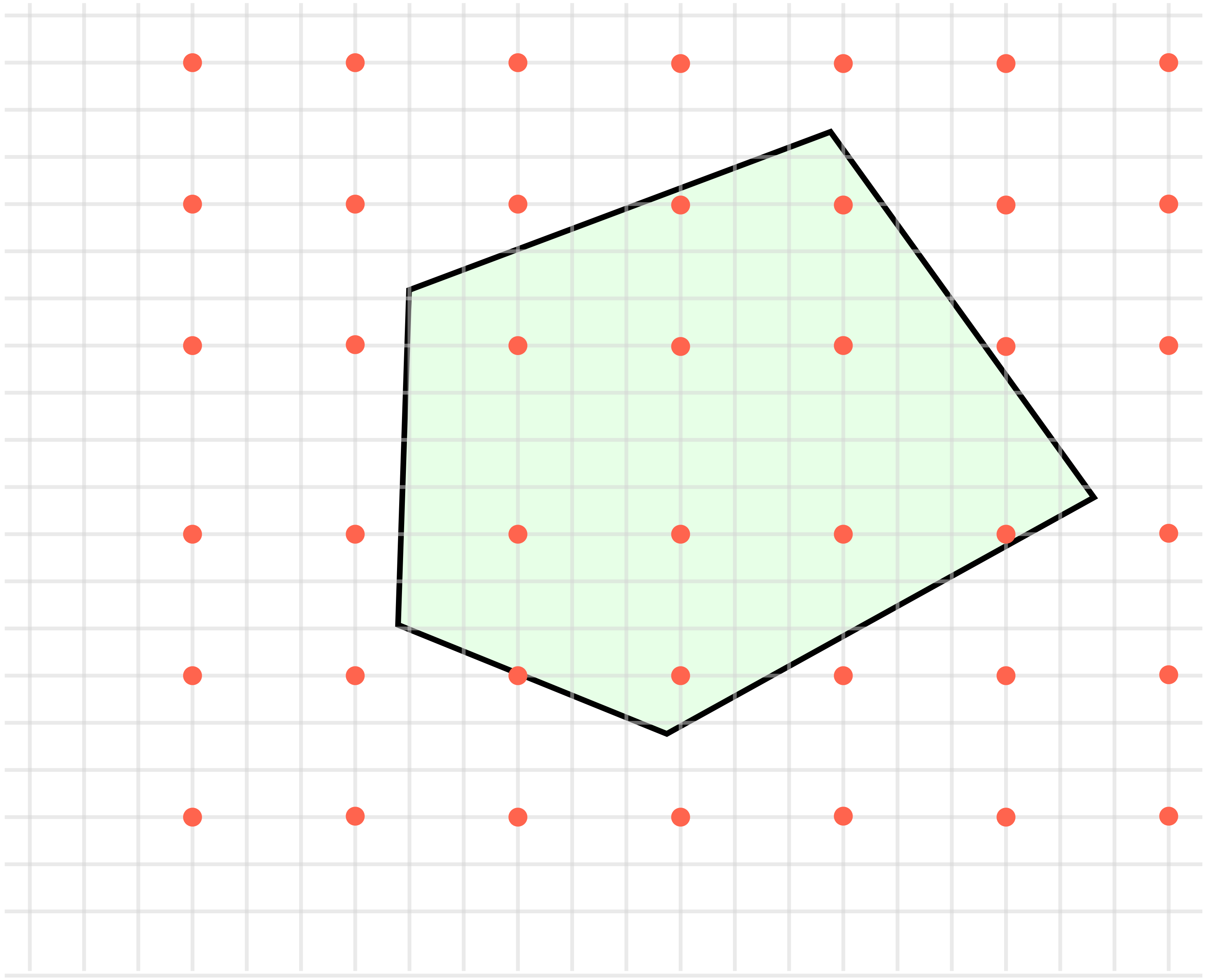

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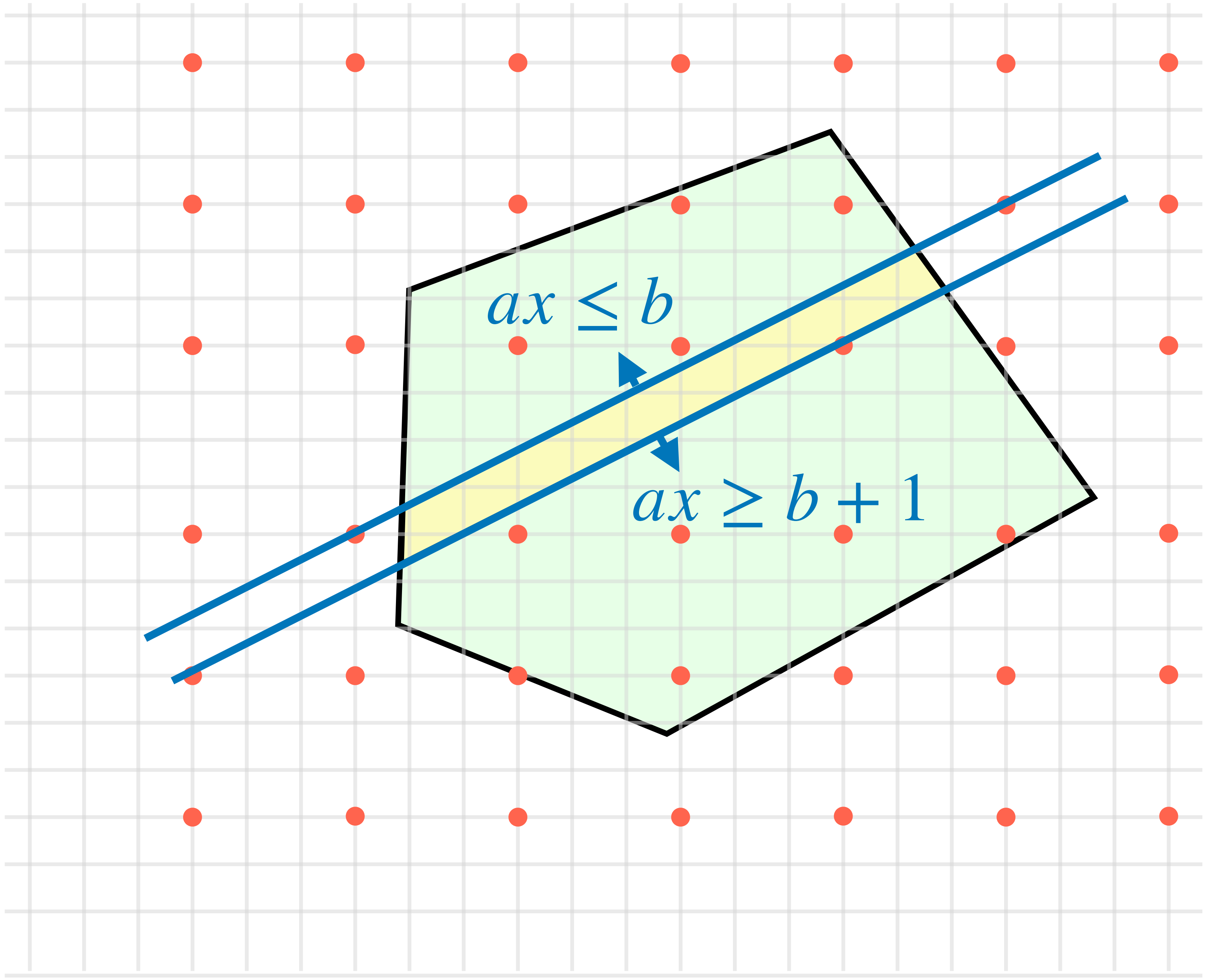

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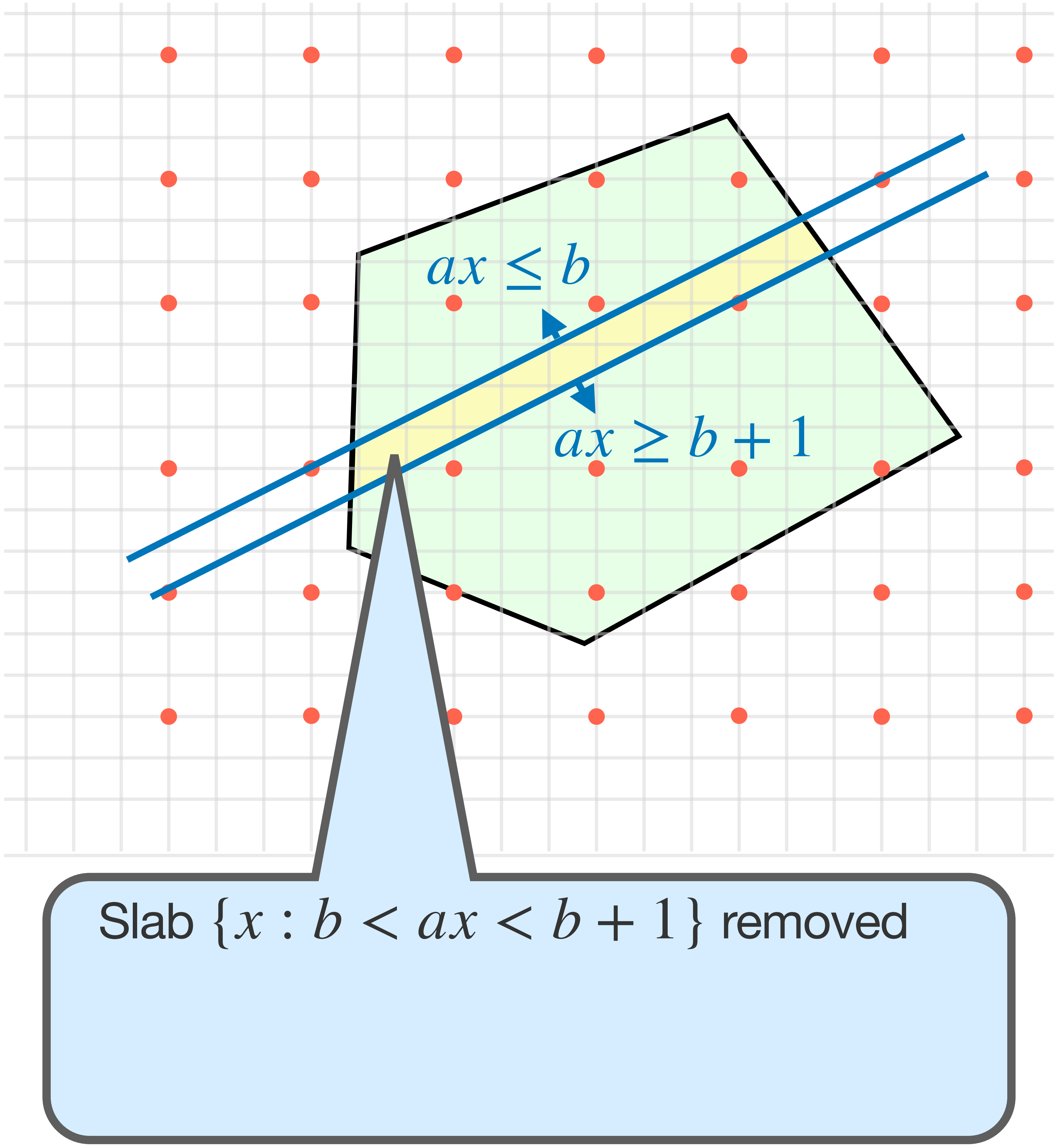

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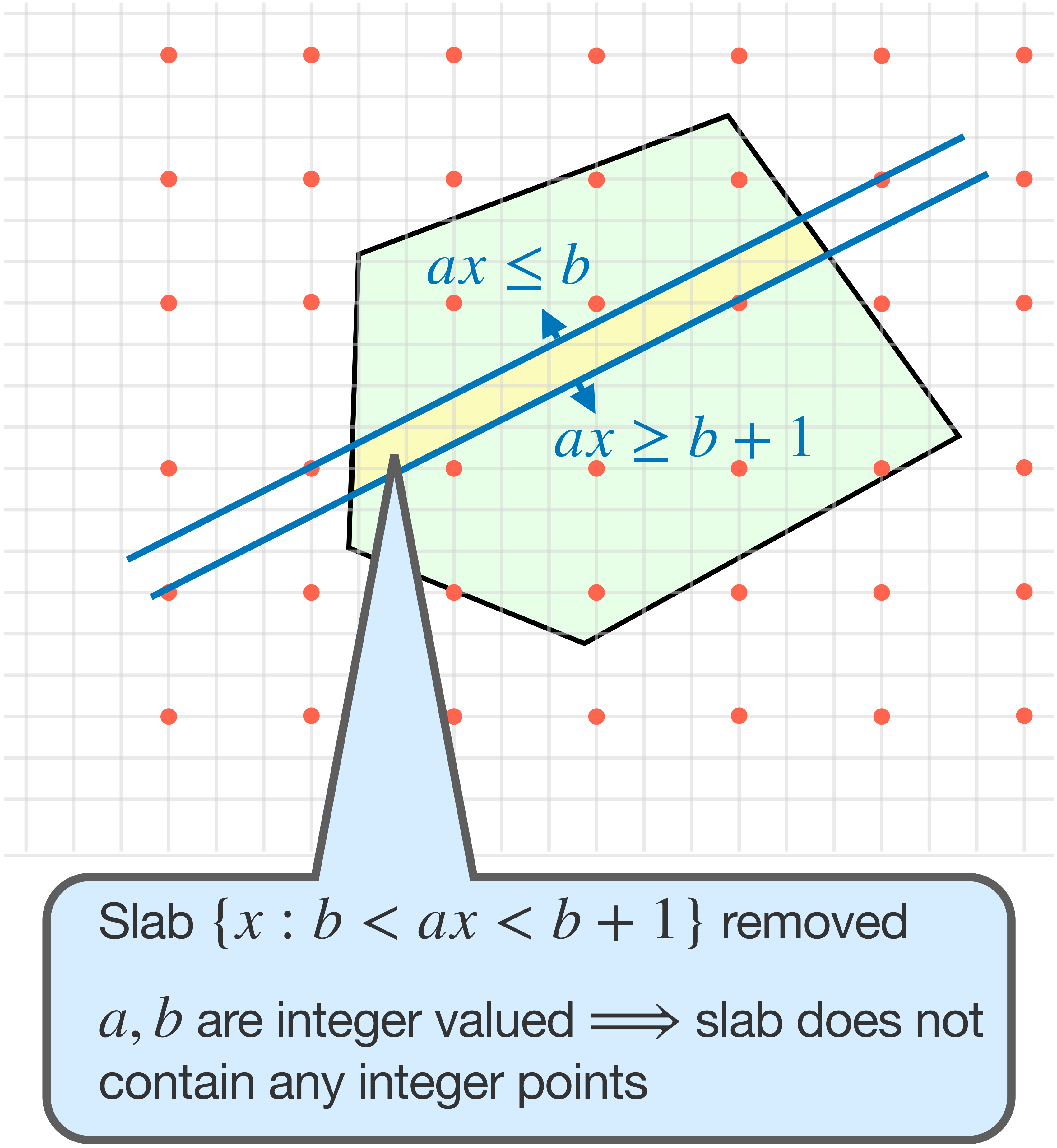

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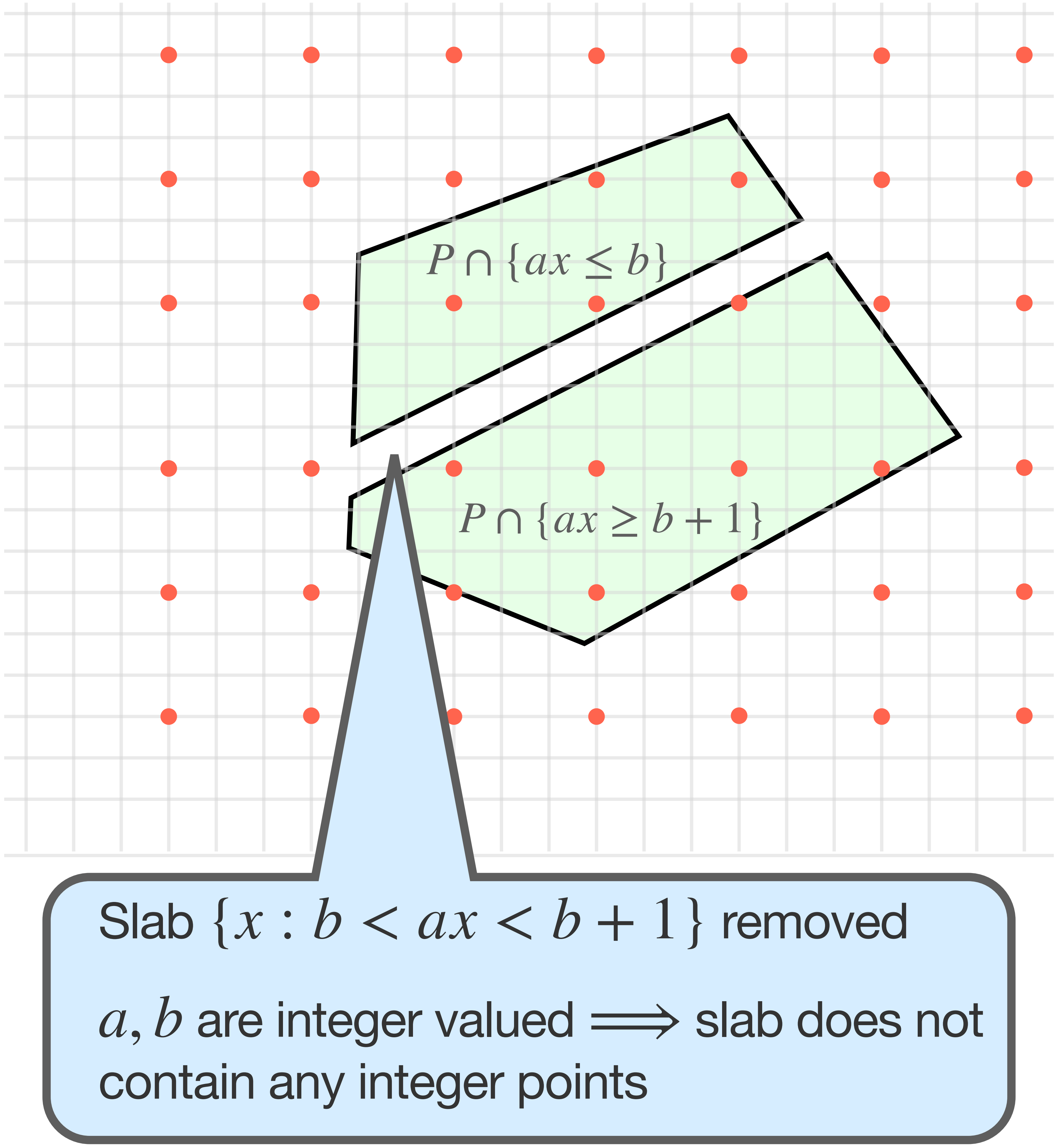

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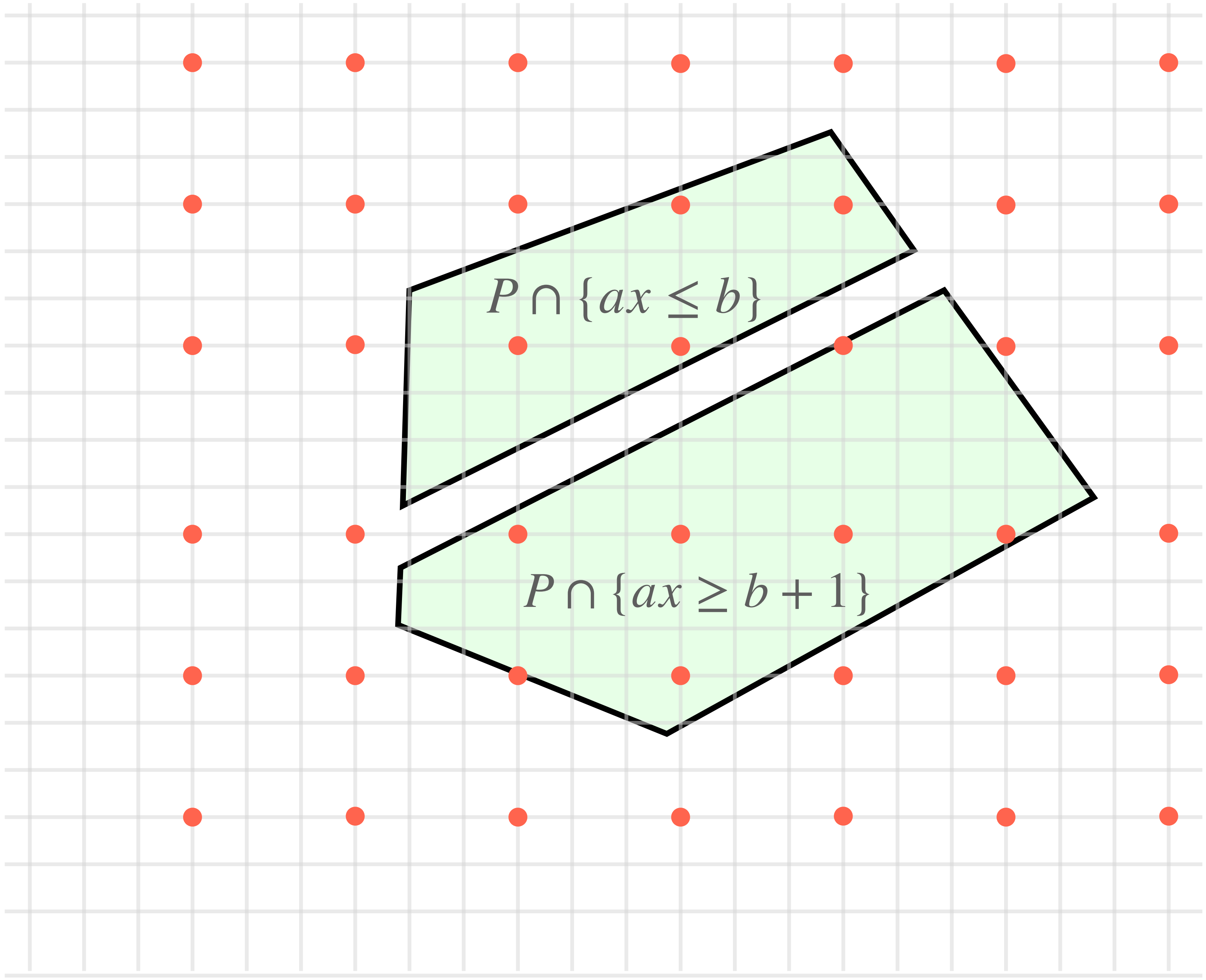

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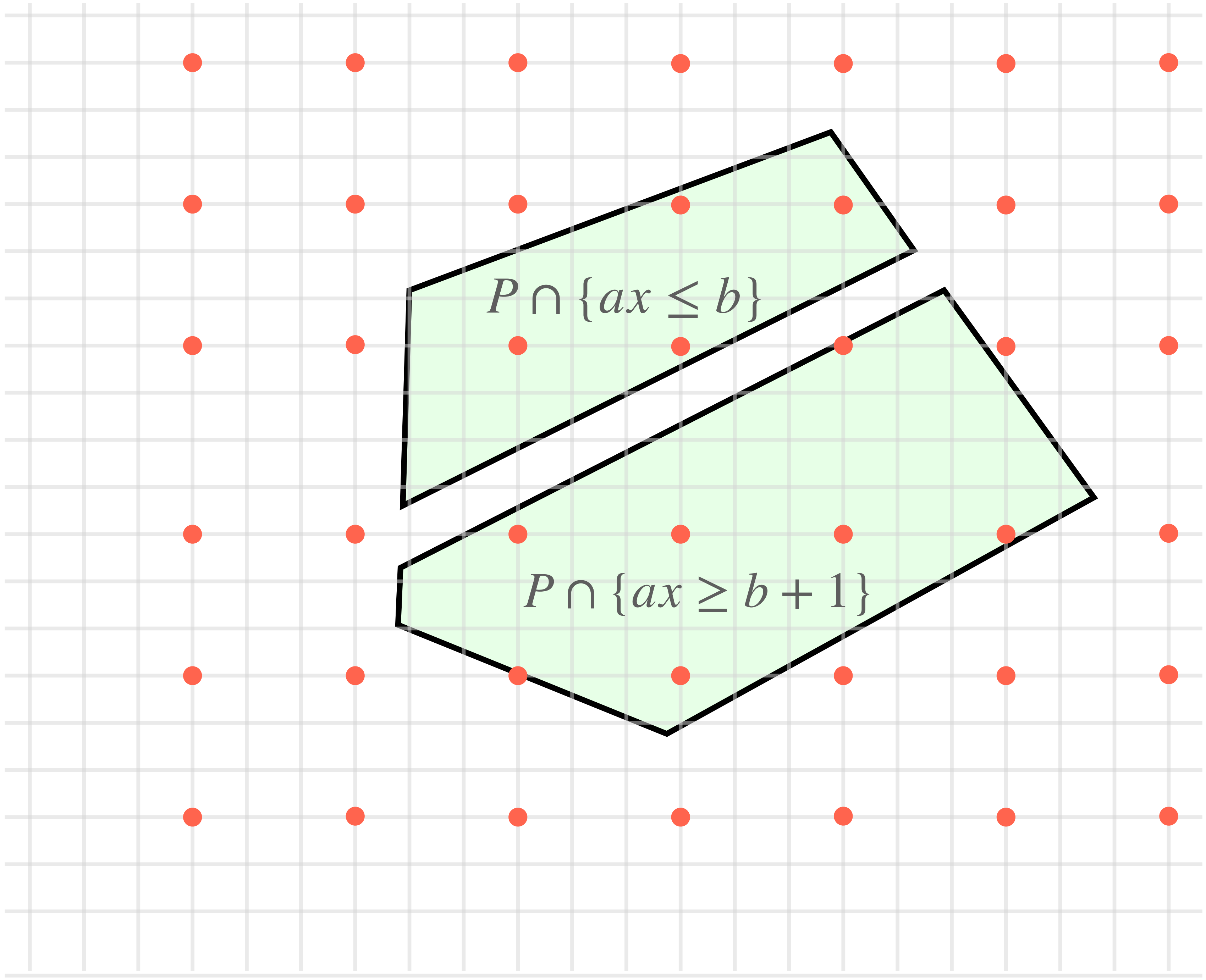
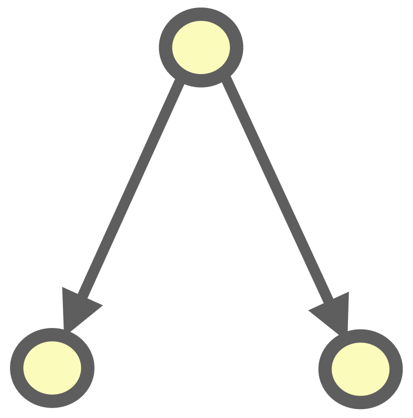


A Stabbing Planes refutation of P with $P \cap \mathbb{Z}^n$ is a binary tree:



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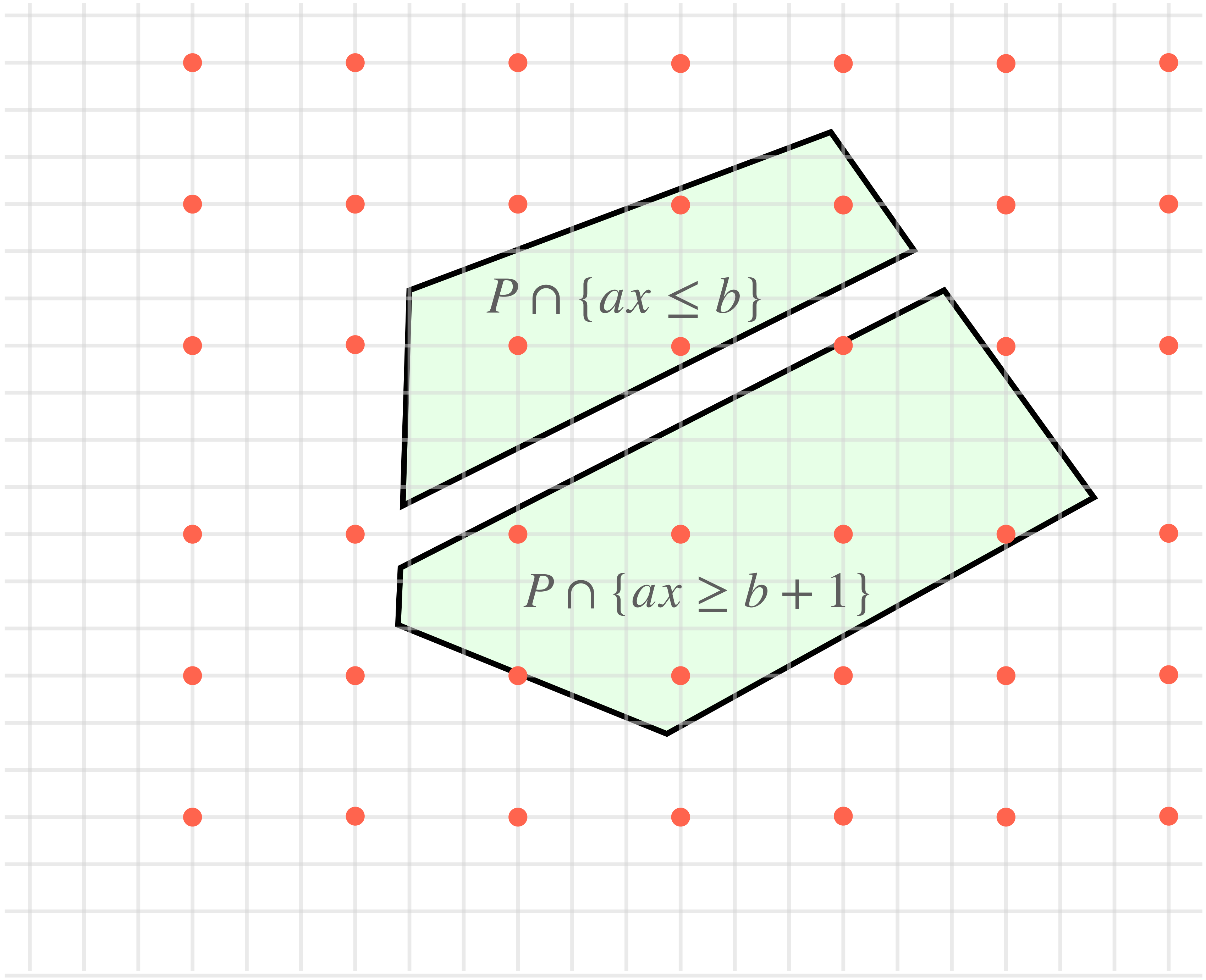

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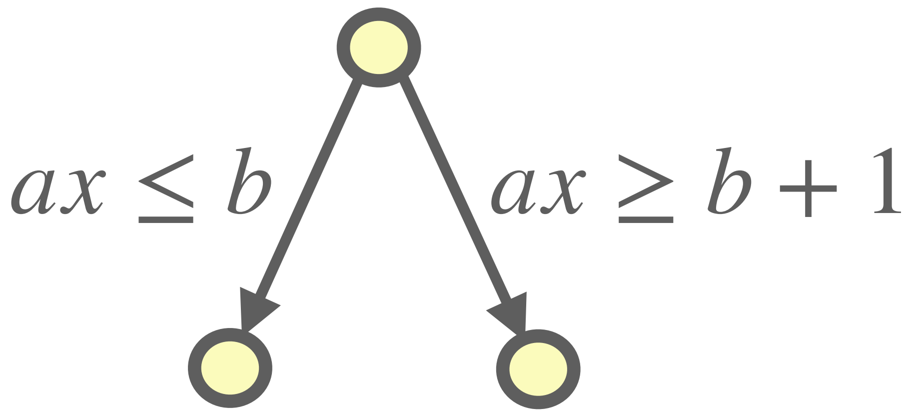

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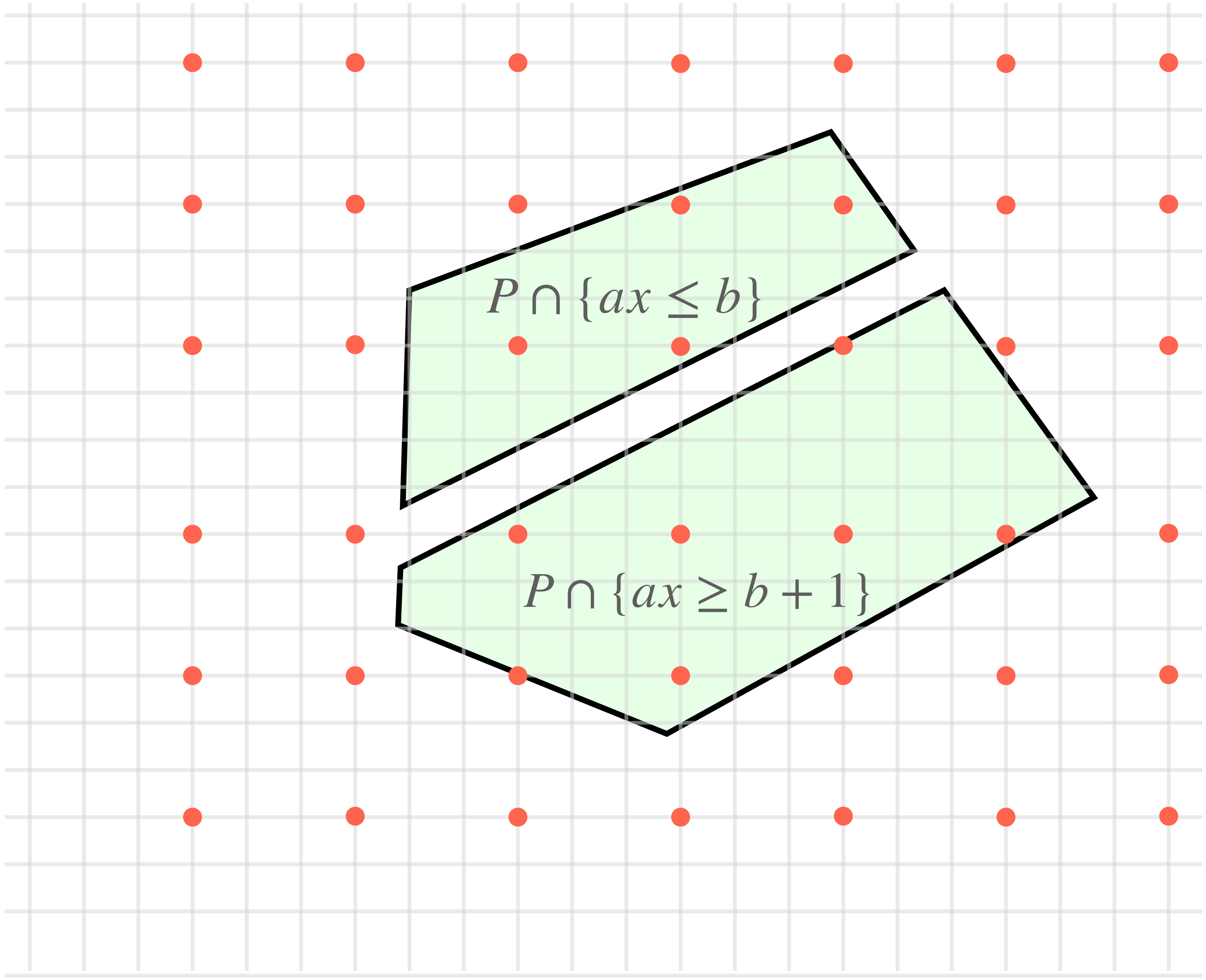

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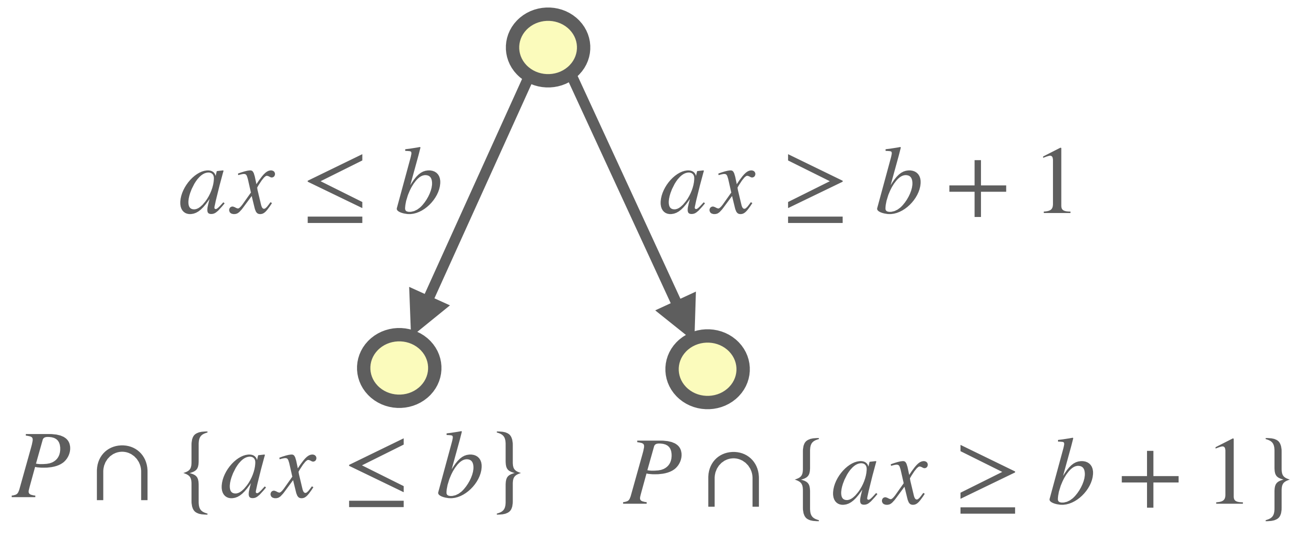

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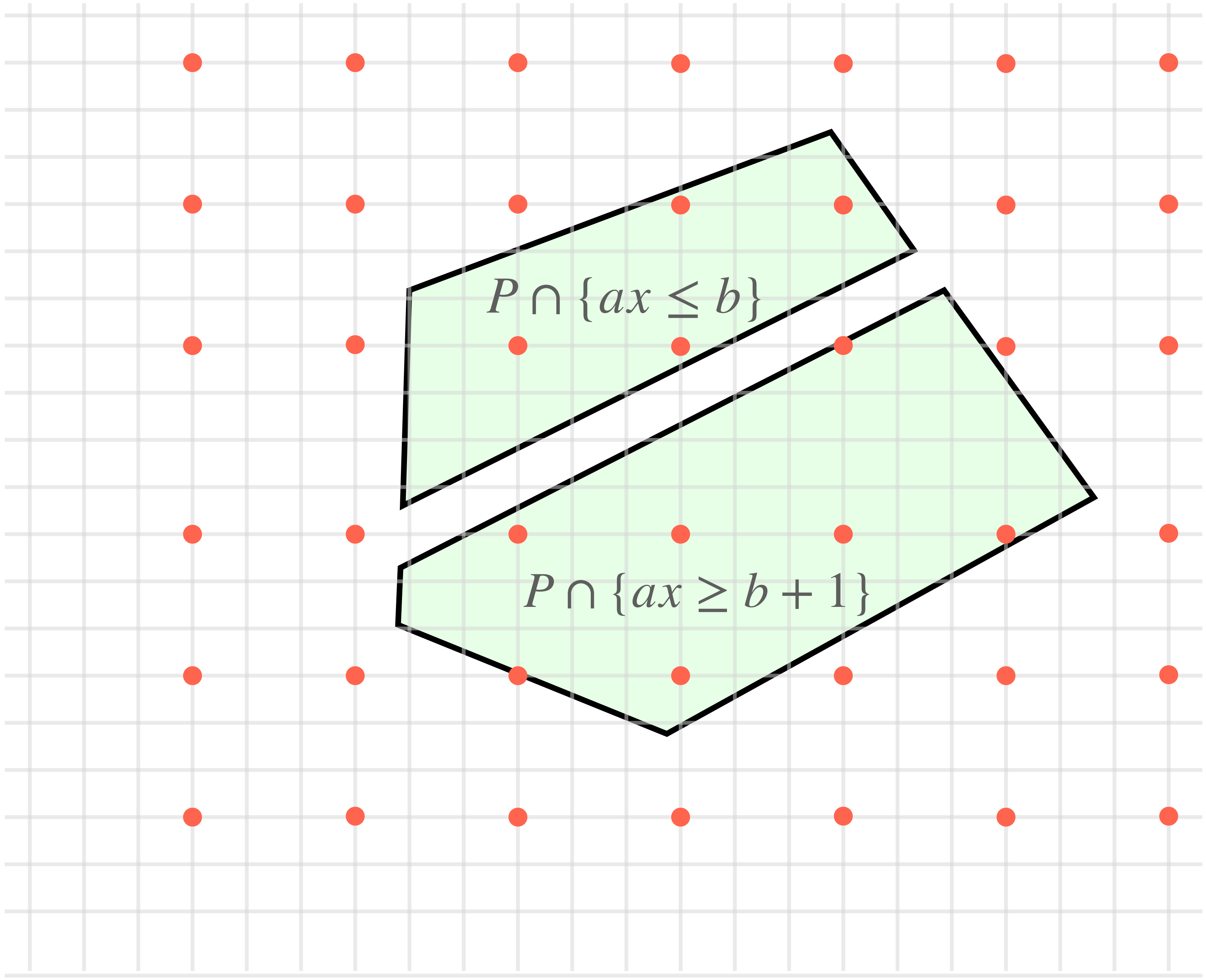

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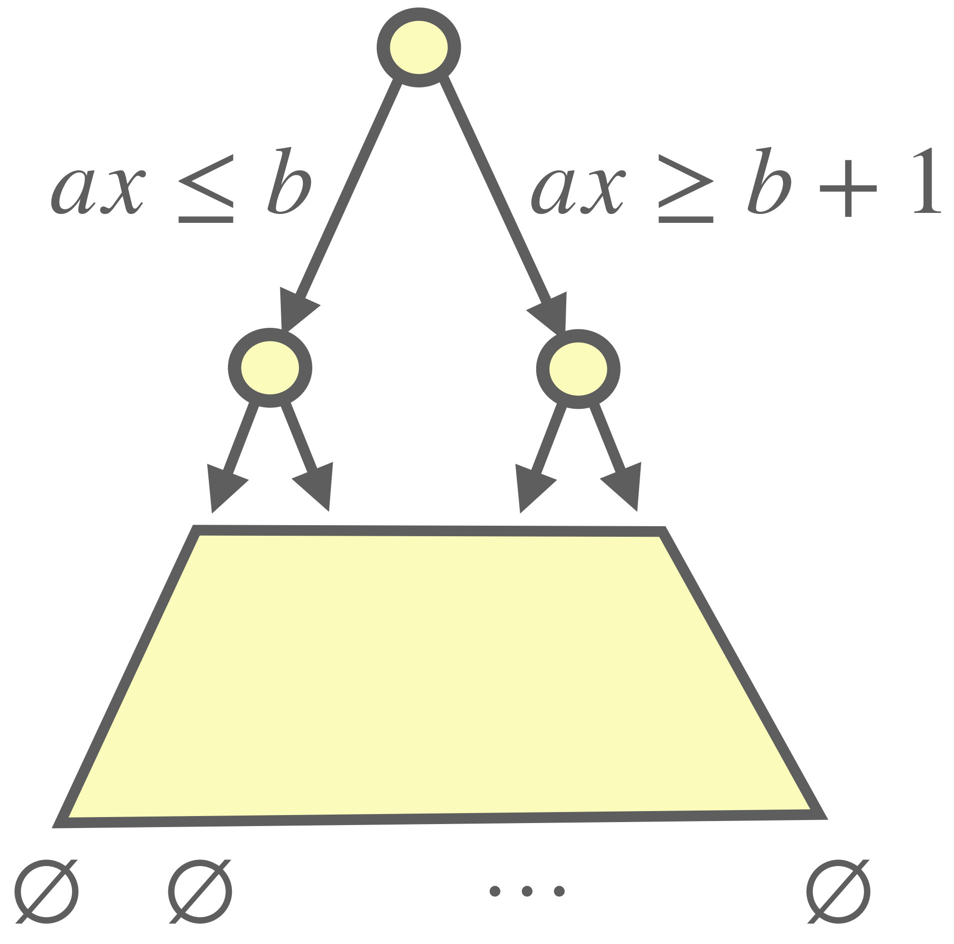

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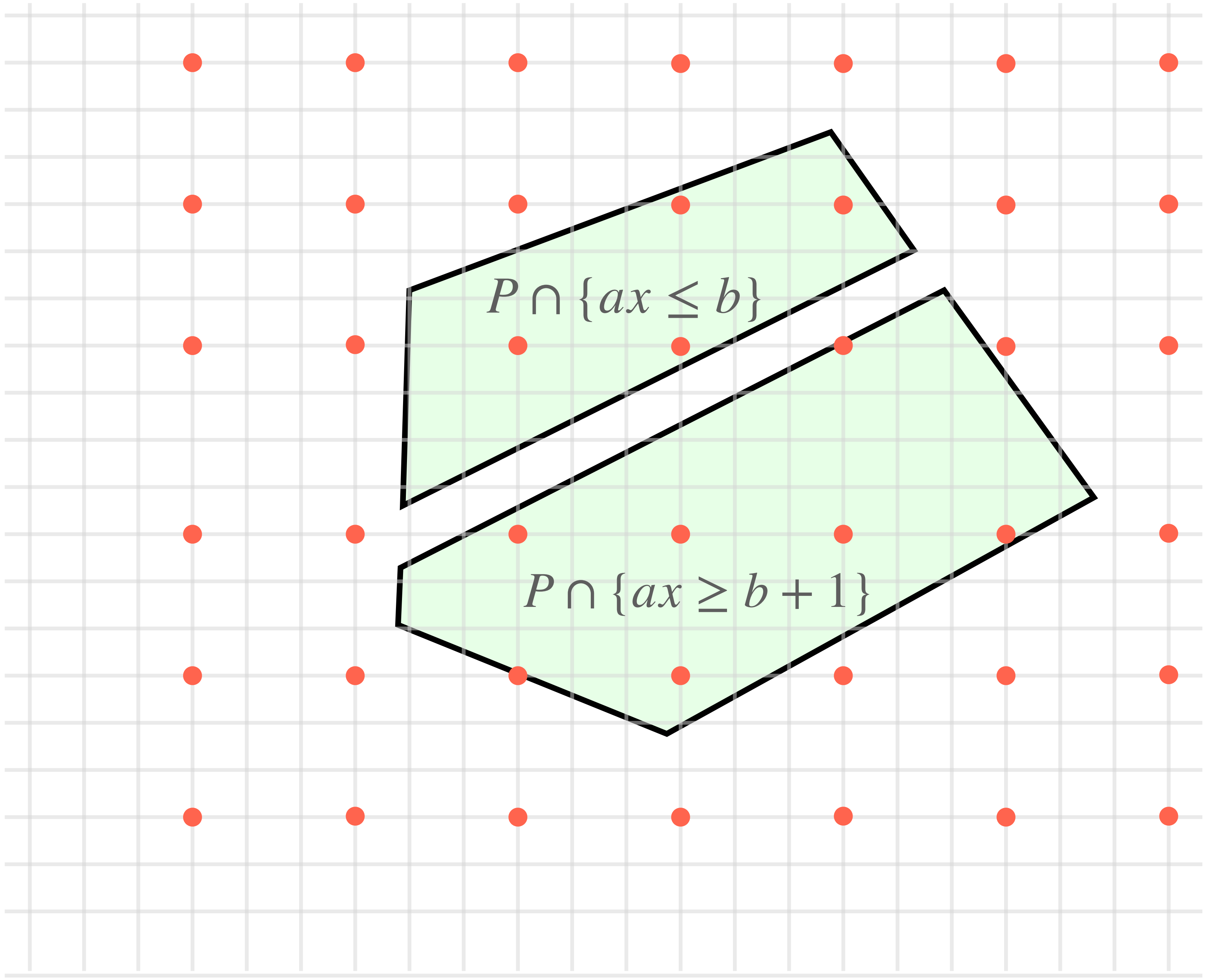

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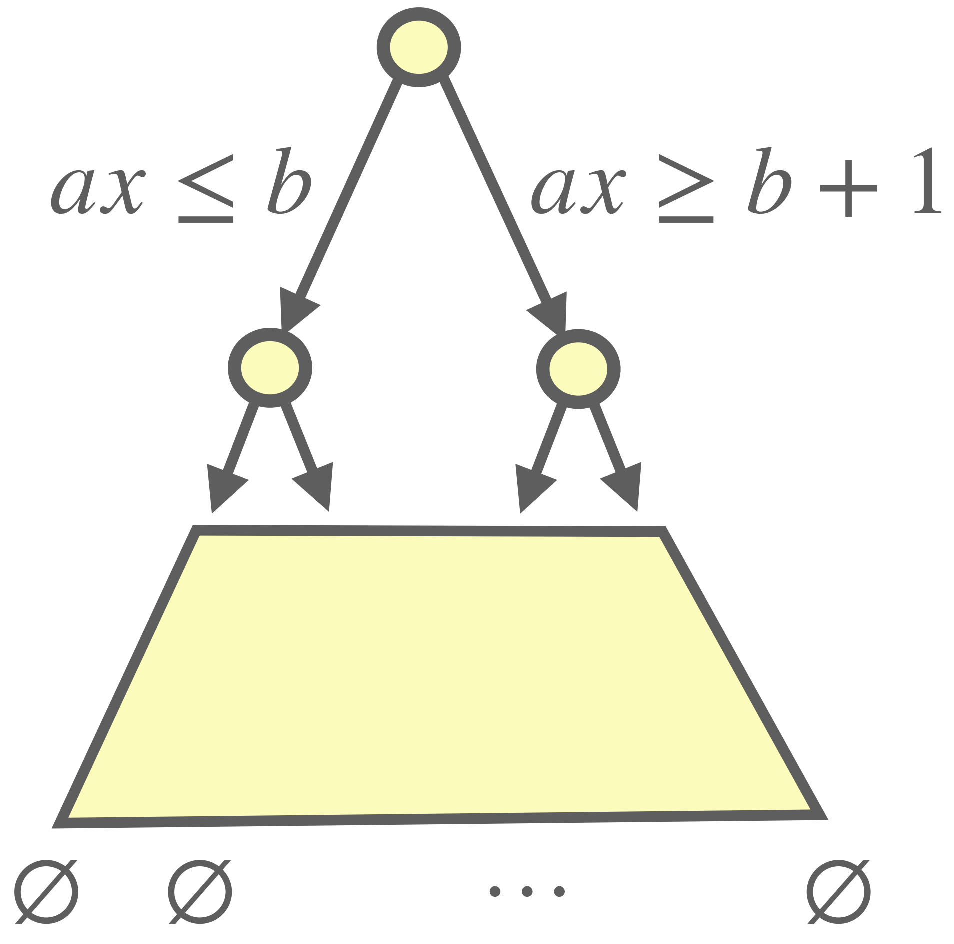
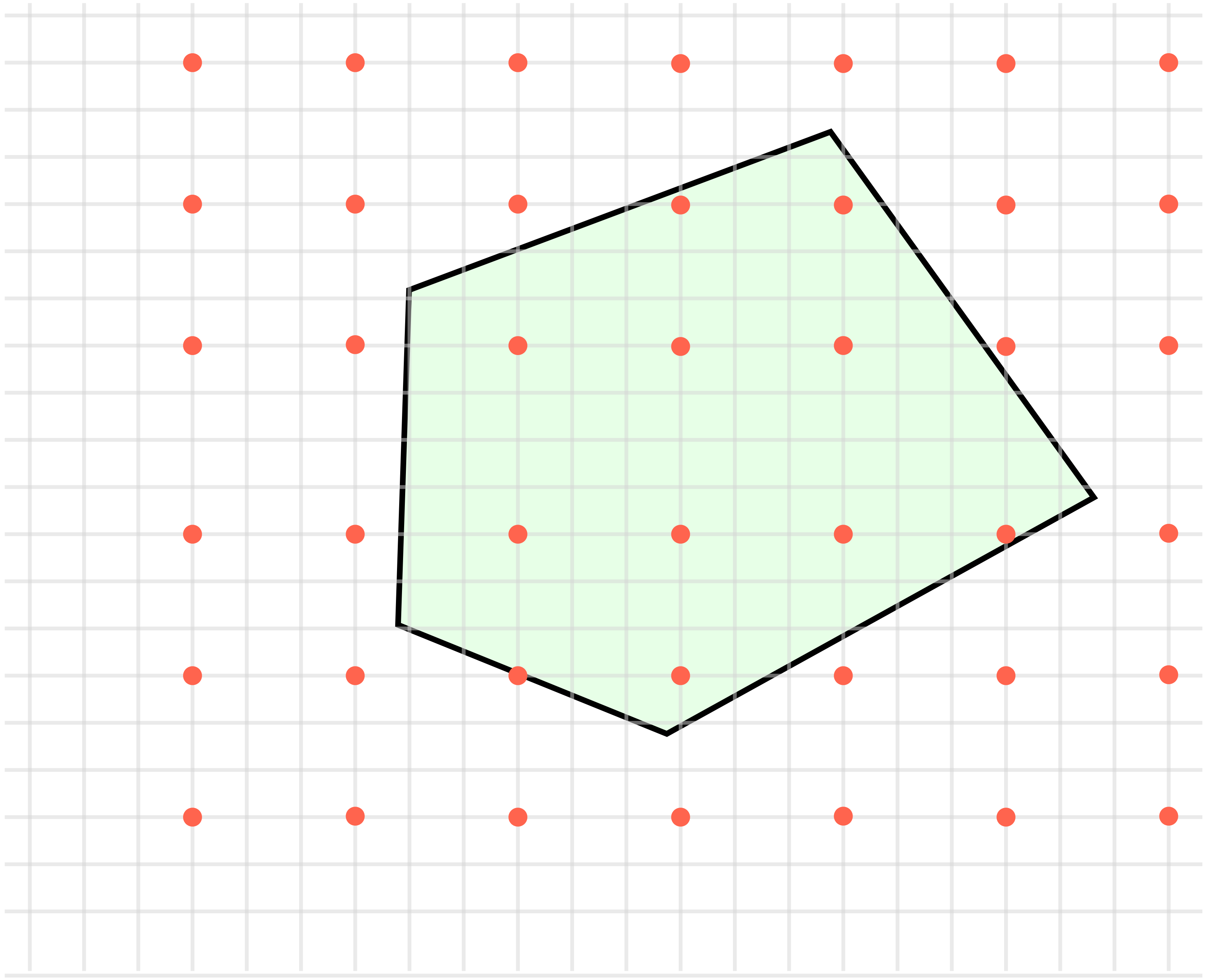


- The size is the number of nodes in the tree

Cutting Planes vs. Stabbing Planes

CP and SP capture separate parts of branch-and-cut



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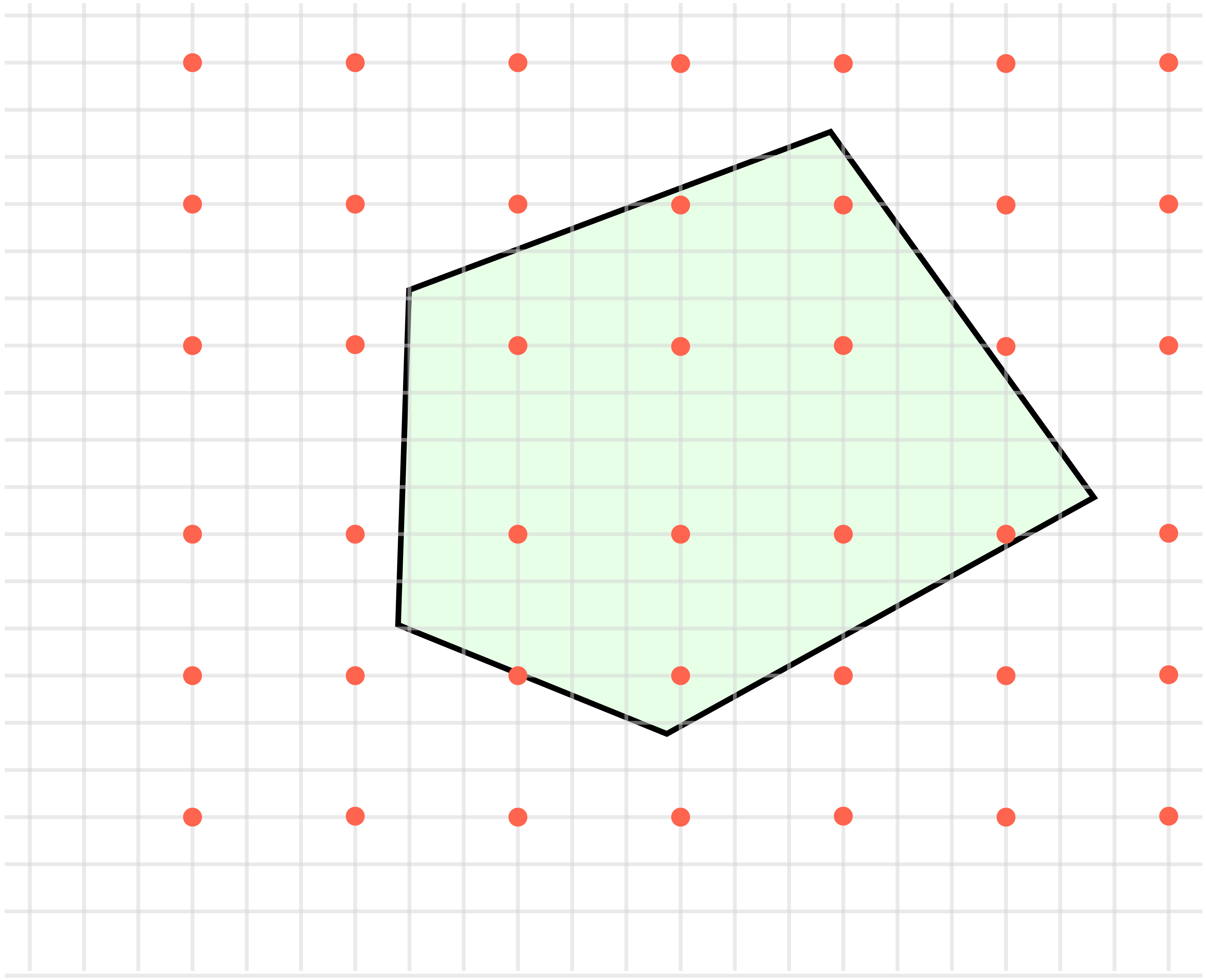

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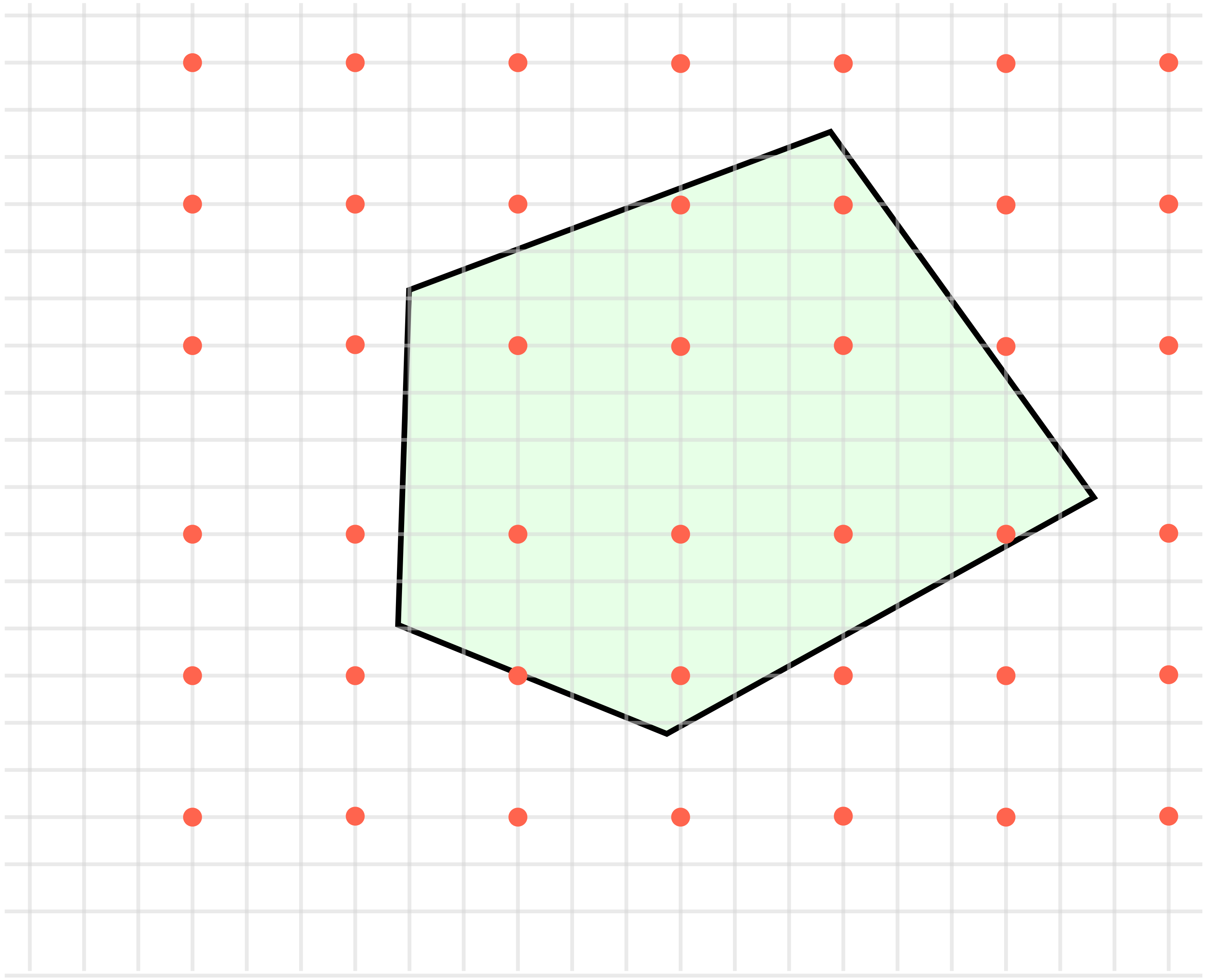

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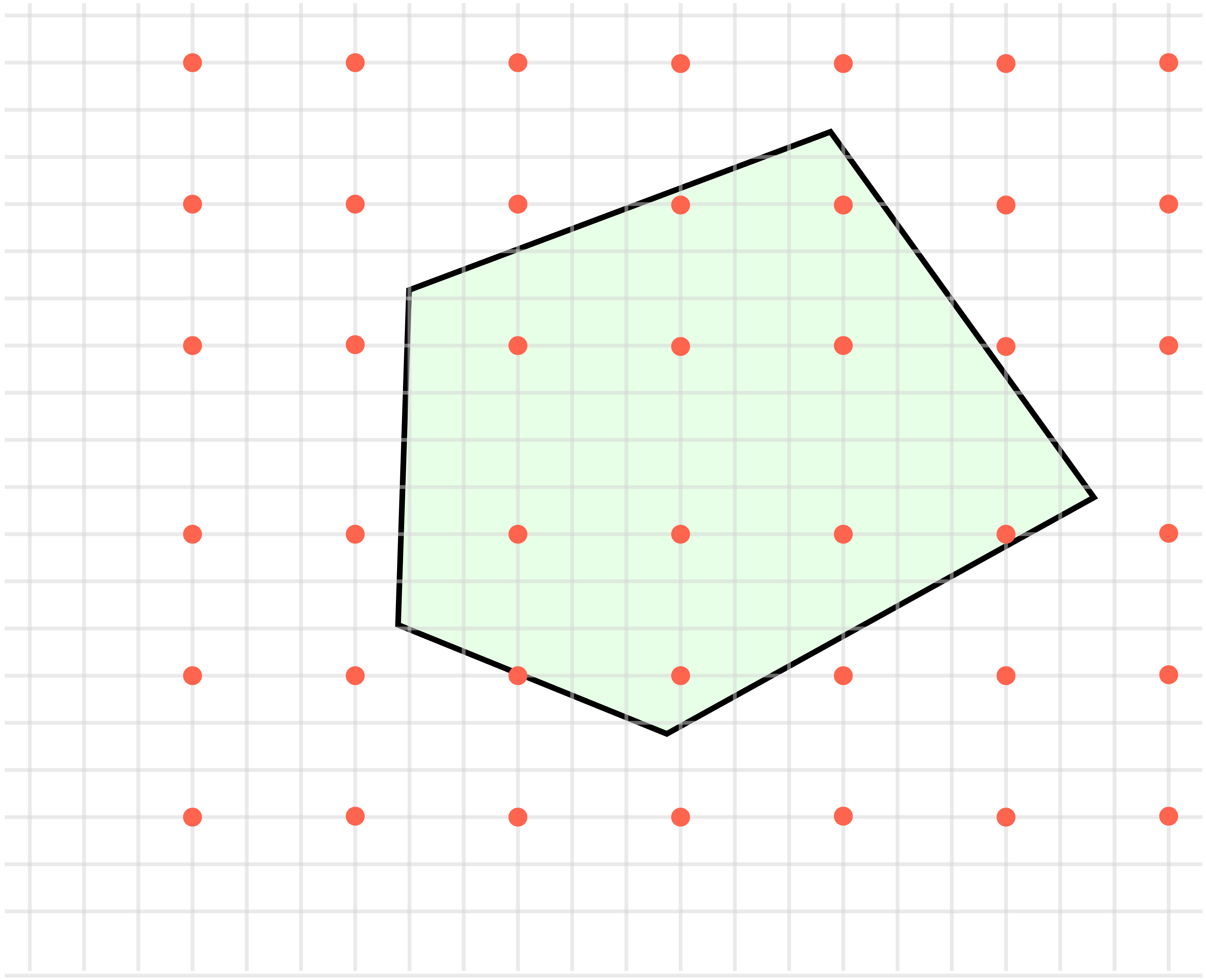

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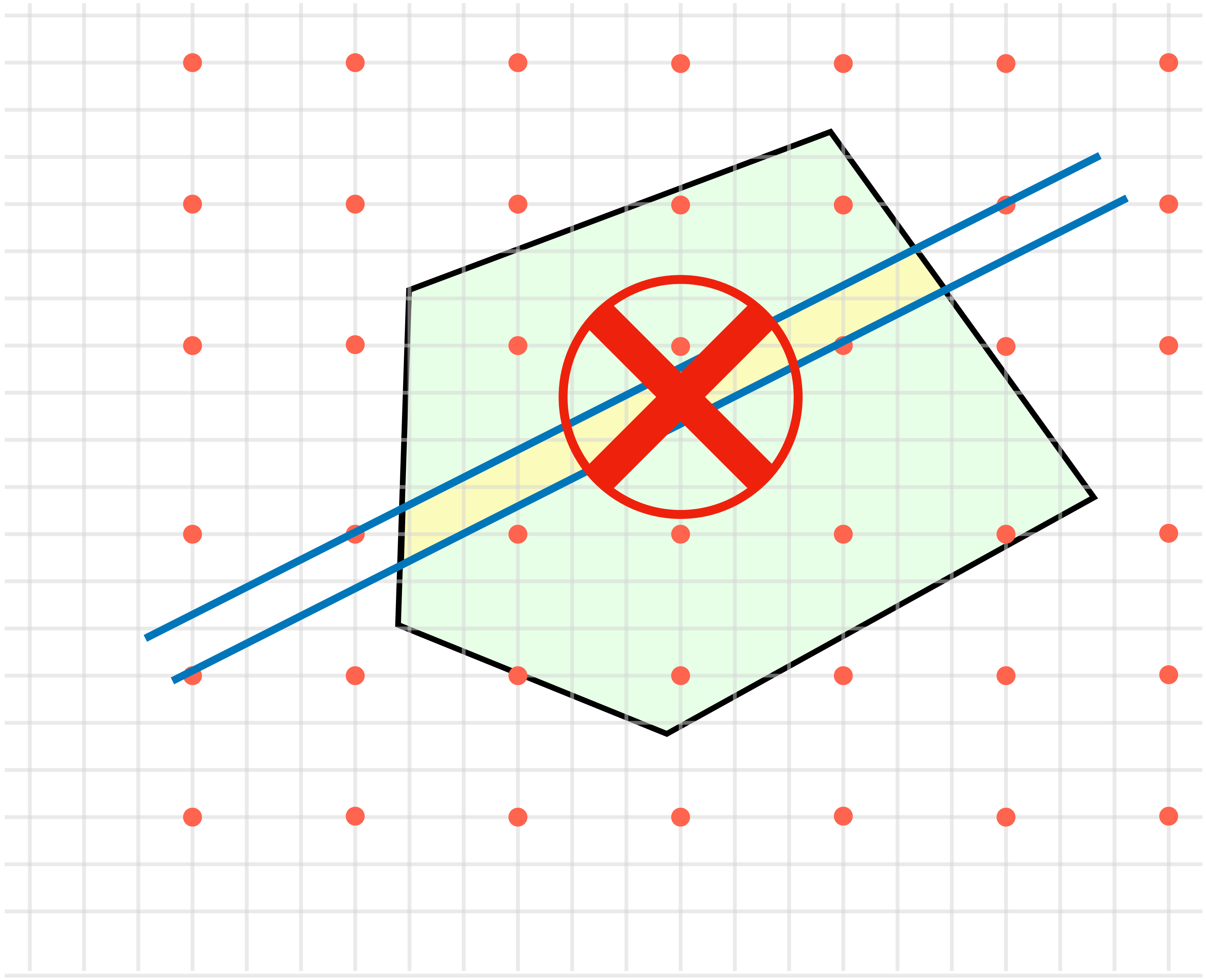

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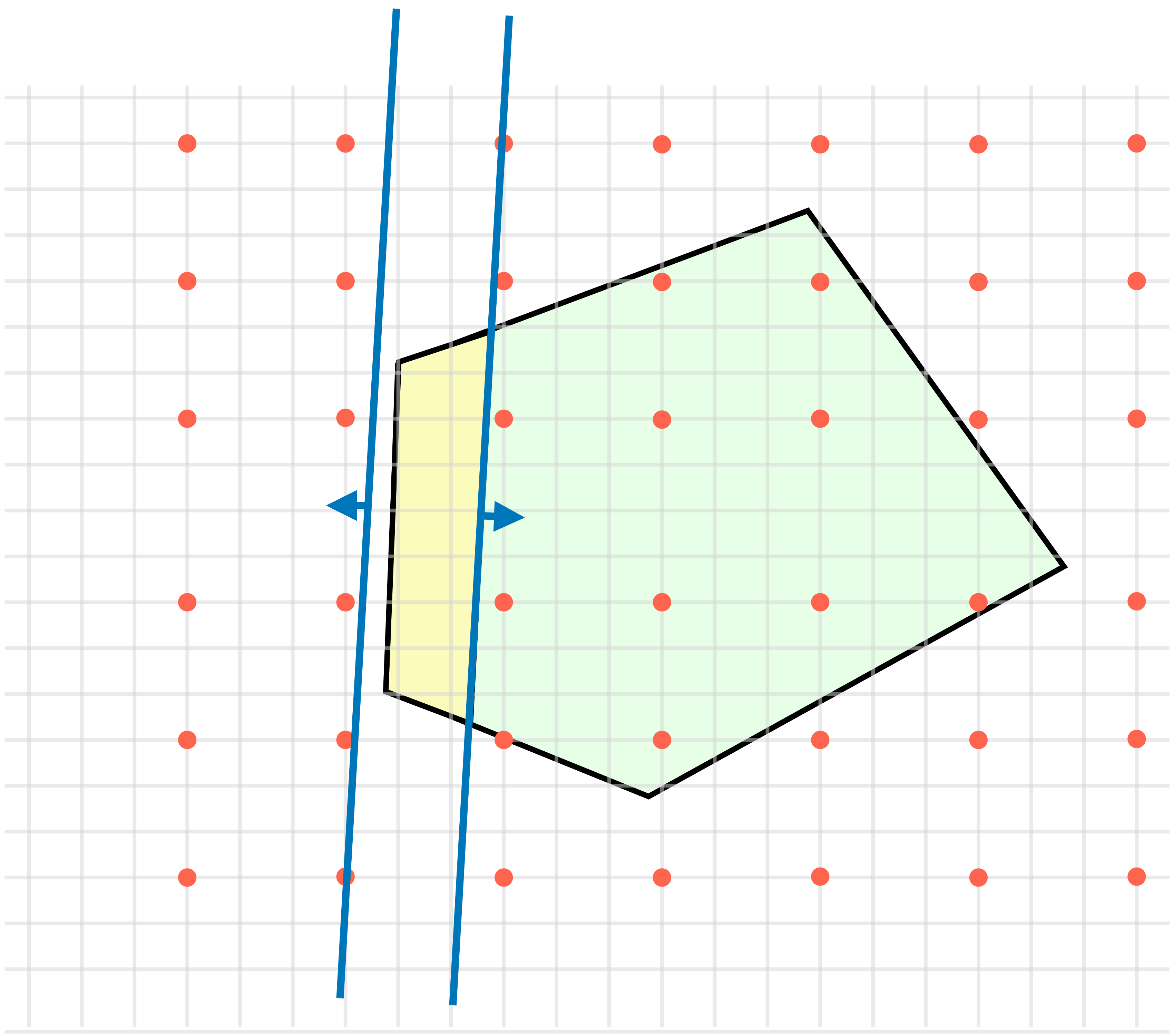

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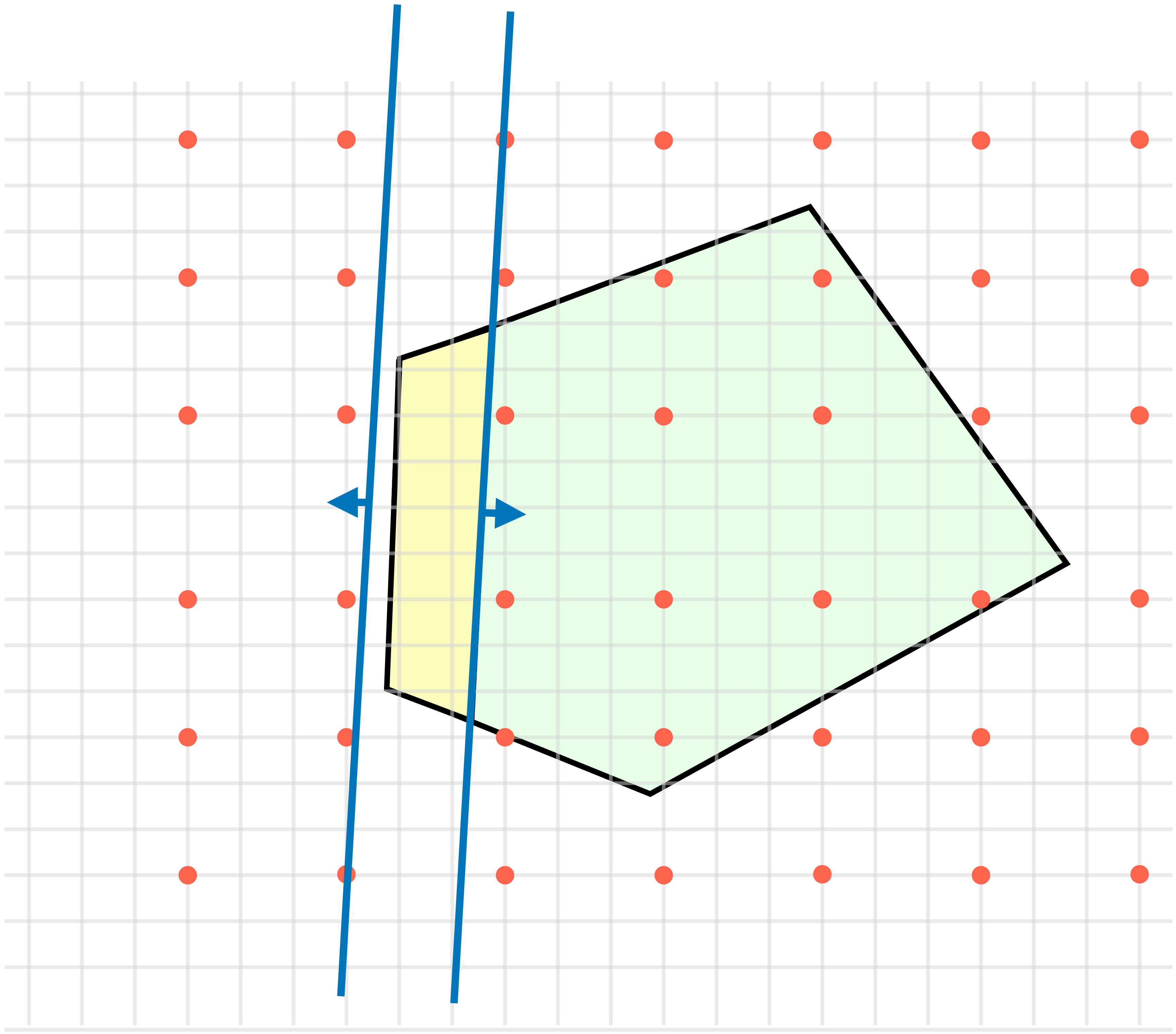

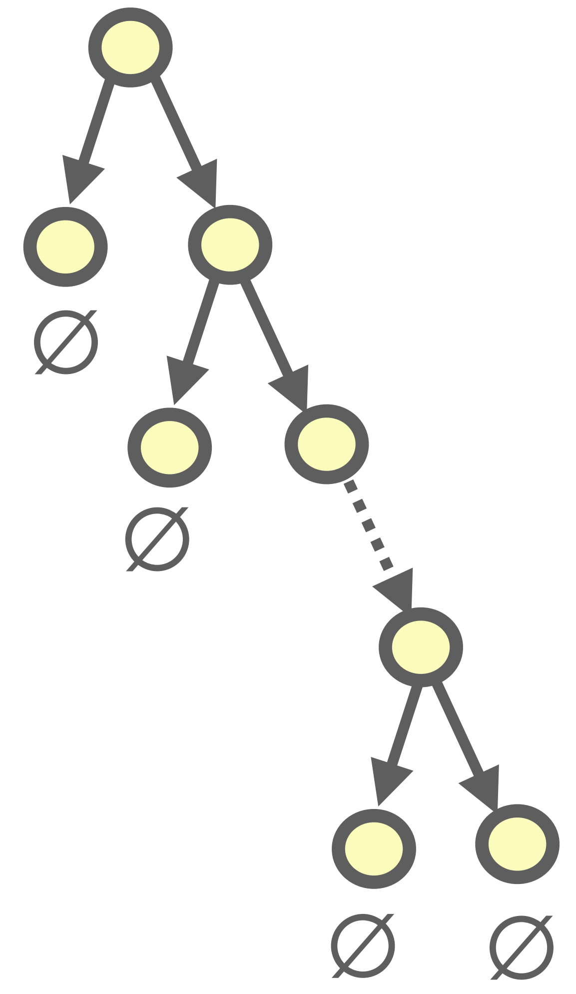

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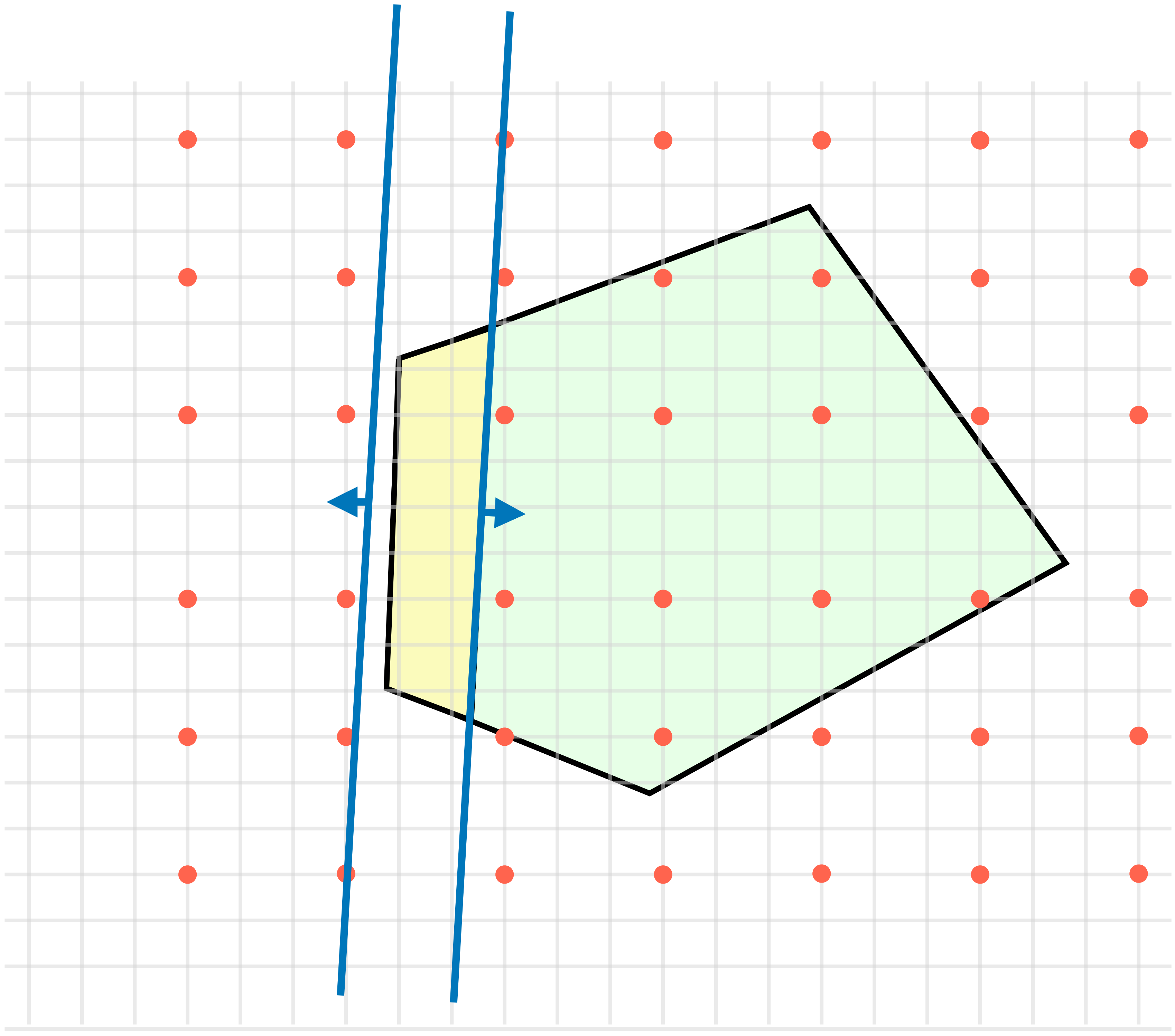
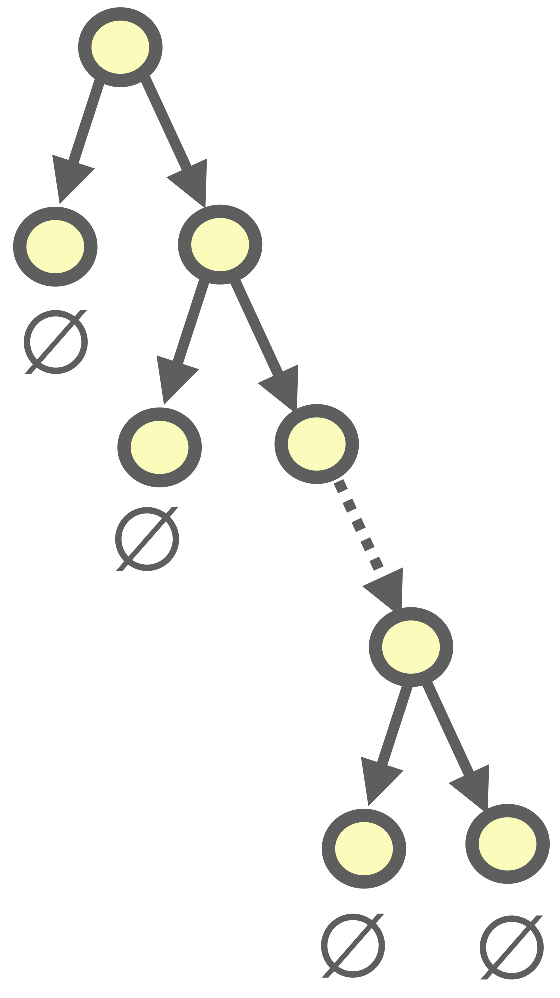
Theorem: Pathlike Stabbing Planes = Cutting Planes





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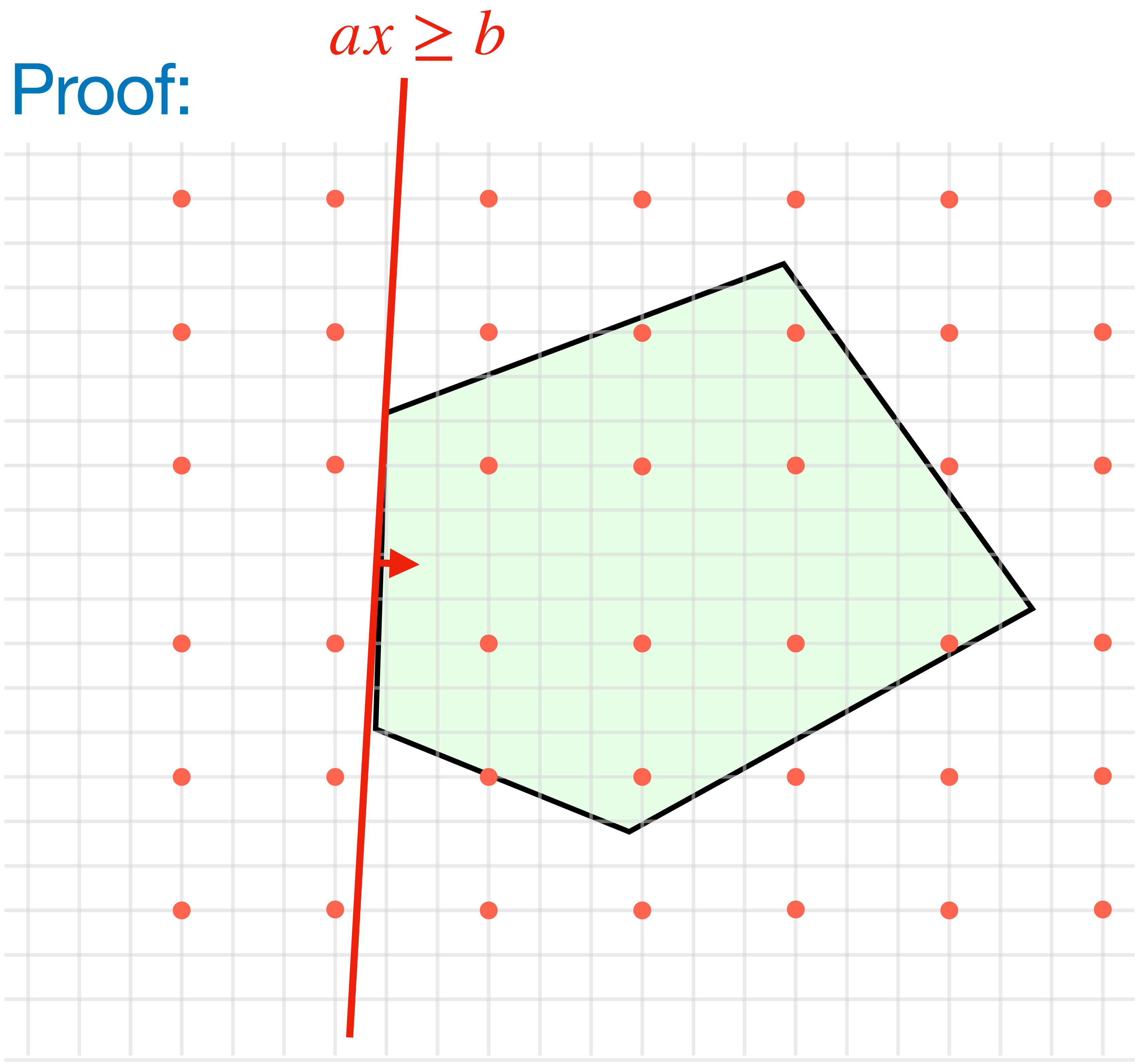

Cutting Planes vs. Stabbing Planes

CP and SP capture separate parts of branch-and-cut

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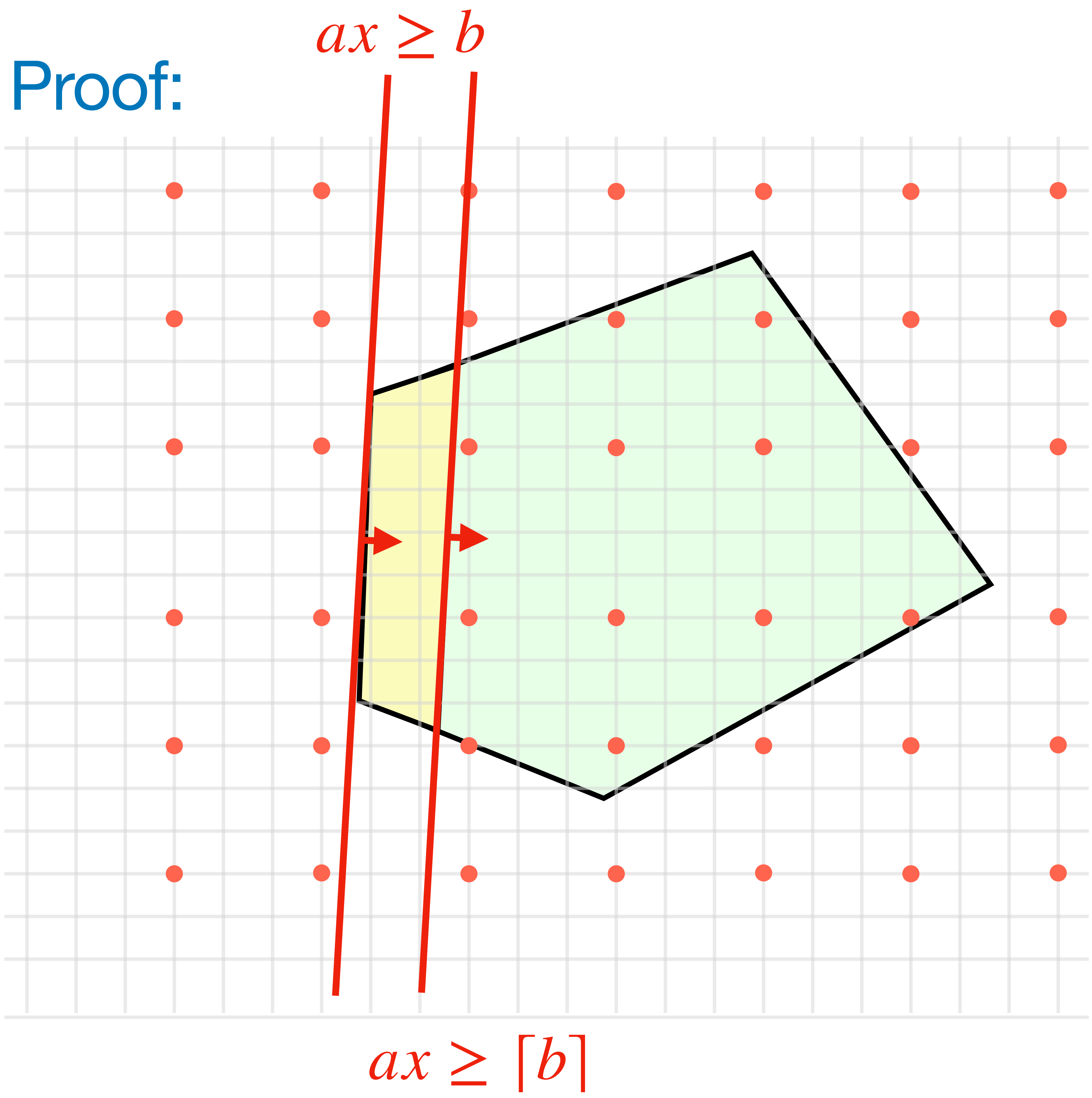

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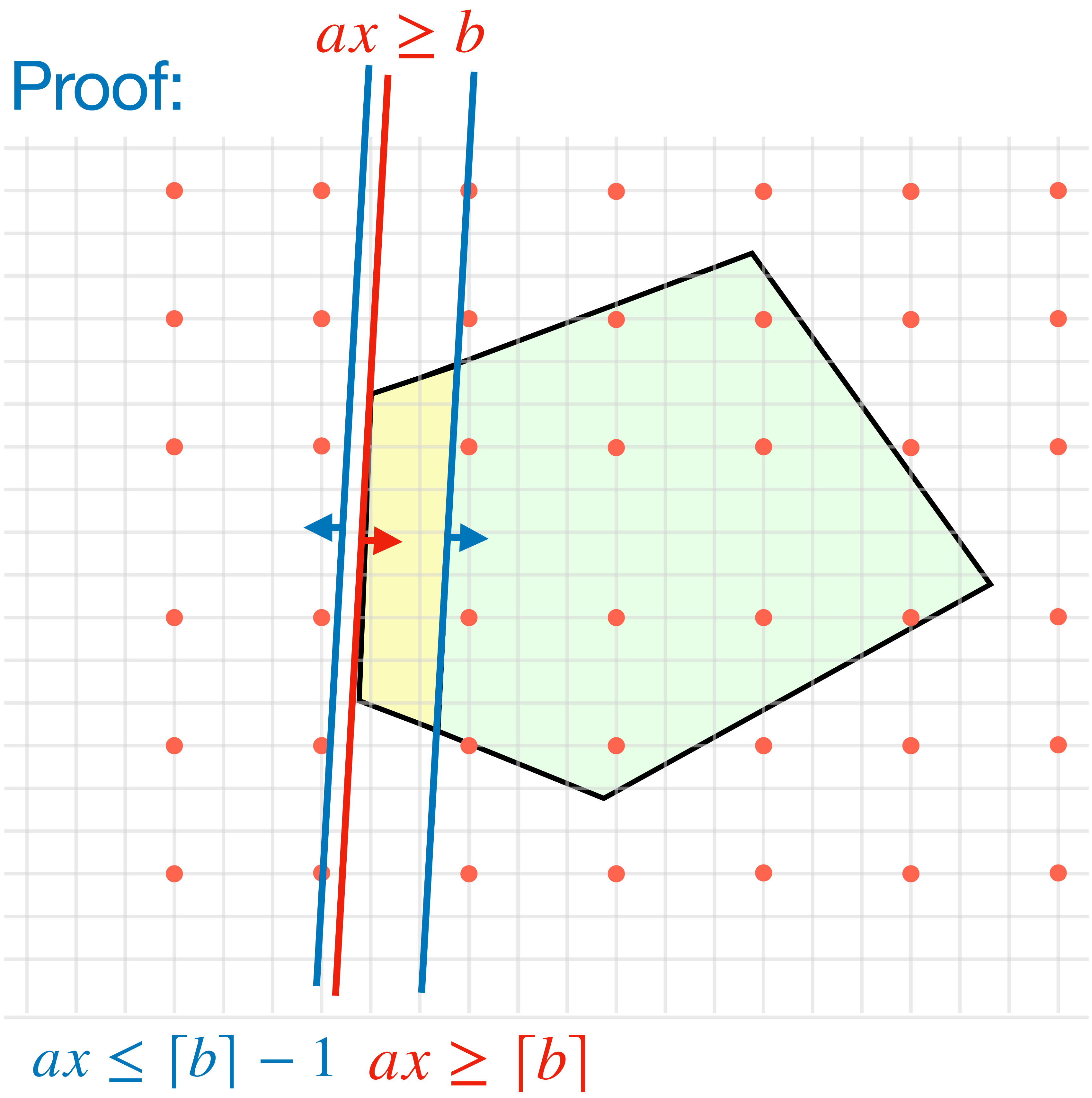

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[DT20] The quasi-polynomial size Stabbing Planes proofs of Tseitin can be translated into quasi-polynomial size Cutting Planes proofs!

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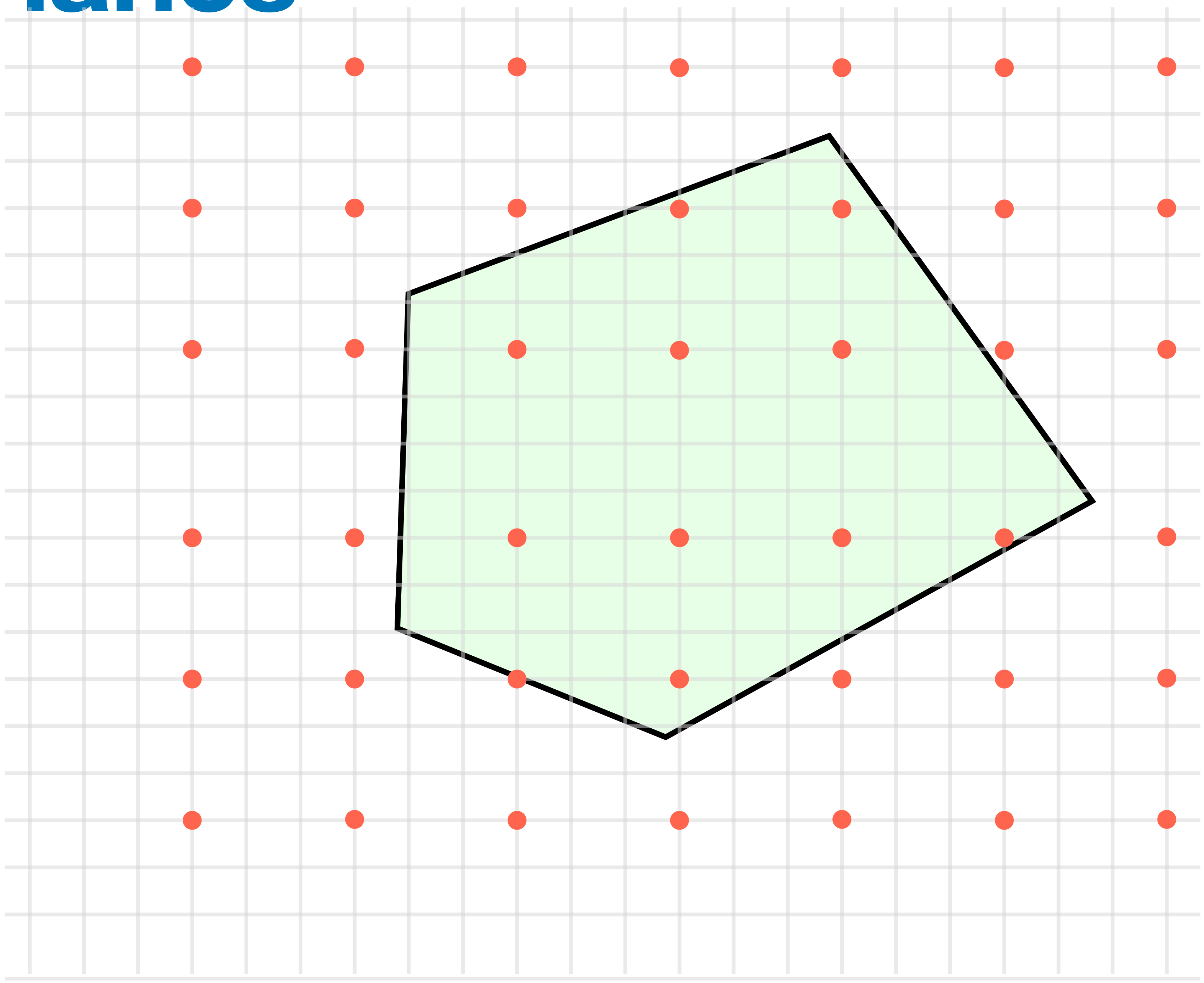
Corollary: Applying existing lower bounds for Cutting Planes proofs [P93, HP17, FPPR17]:

- The clique-colour formulas requires exponential size bounded-coefficient SP proofs.
- Random $\Theta(\log n)$-CNF formulas require exponential size bounded-coefficient SP proofs.

Proof Idea: Stabbing Planes* → Cutting Planes

Two steps

1. CP = Pathlike SP



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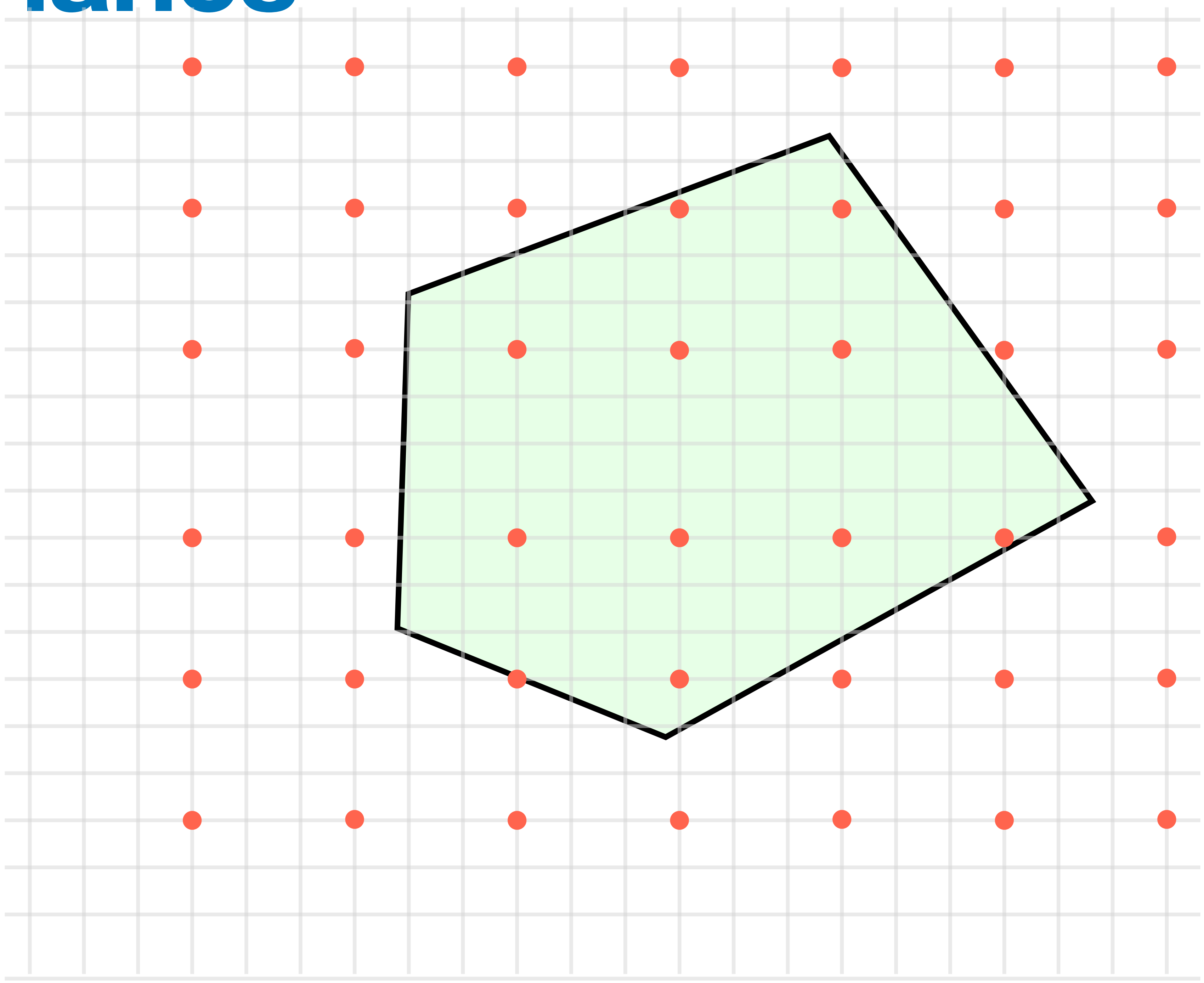

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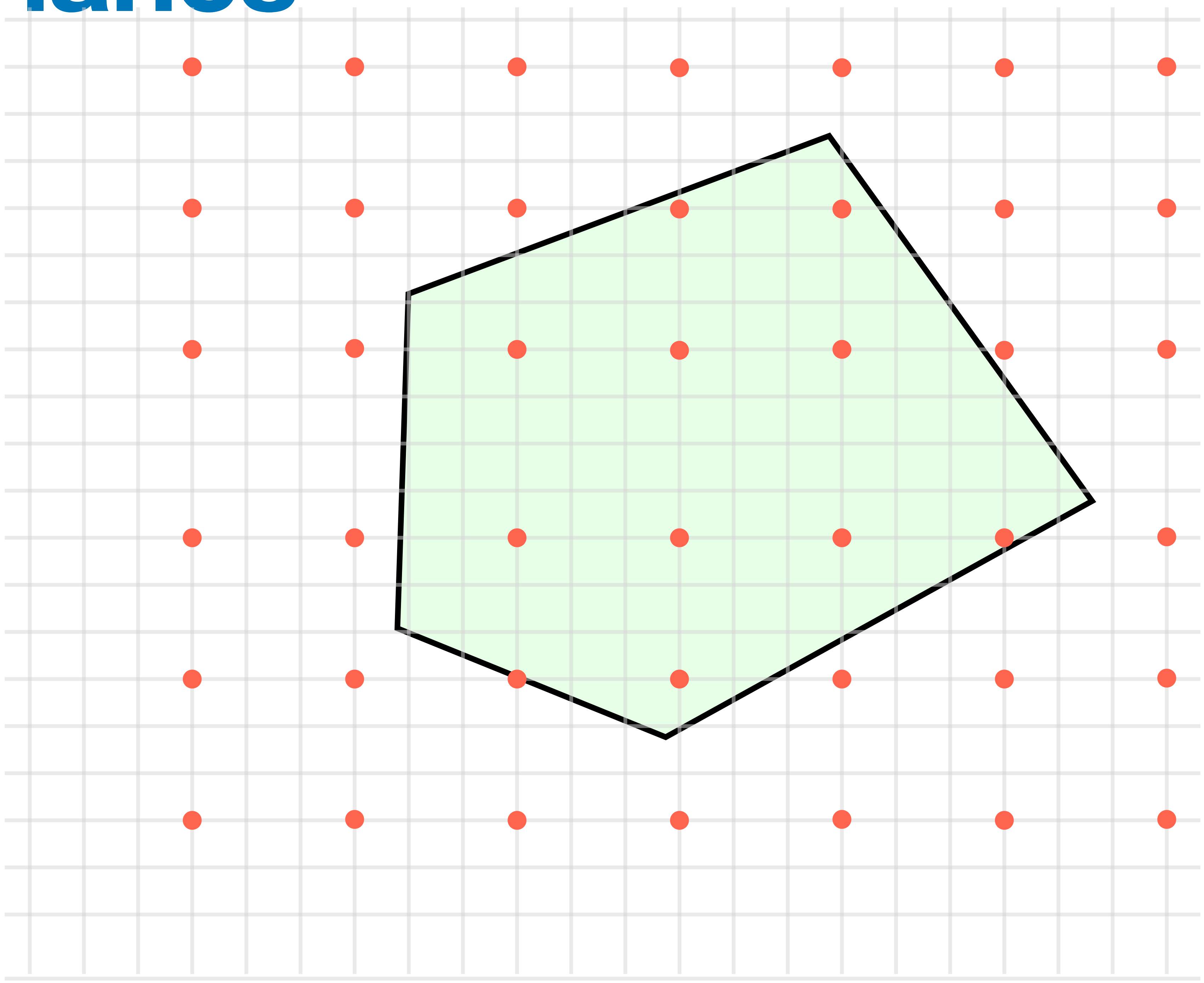

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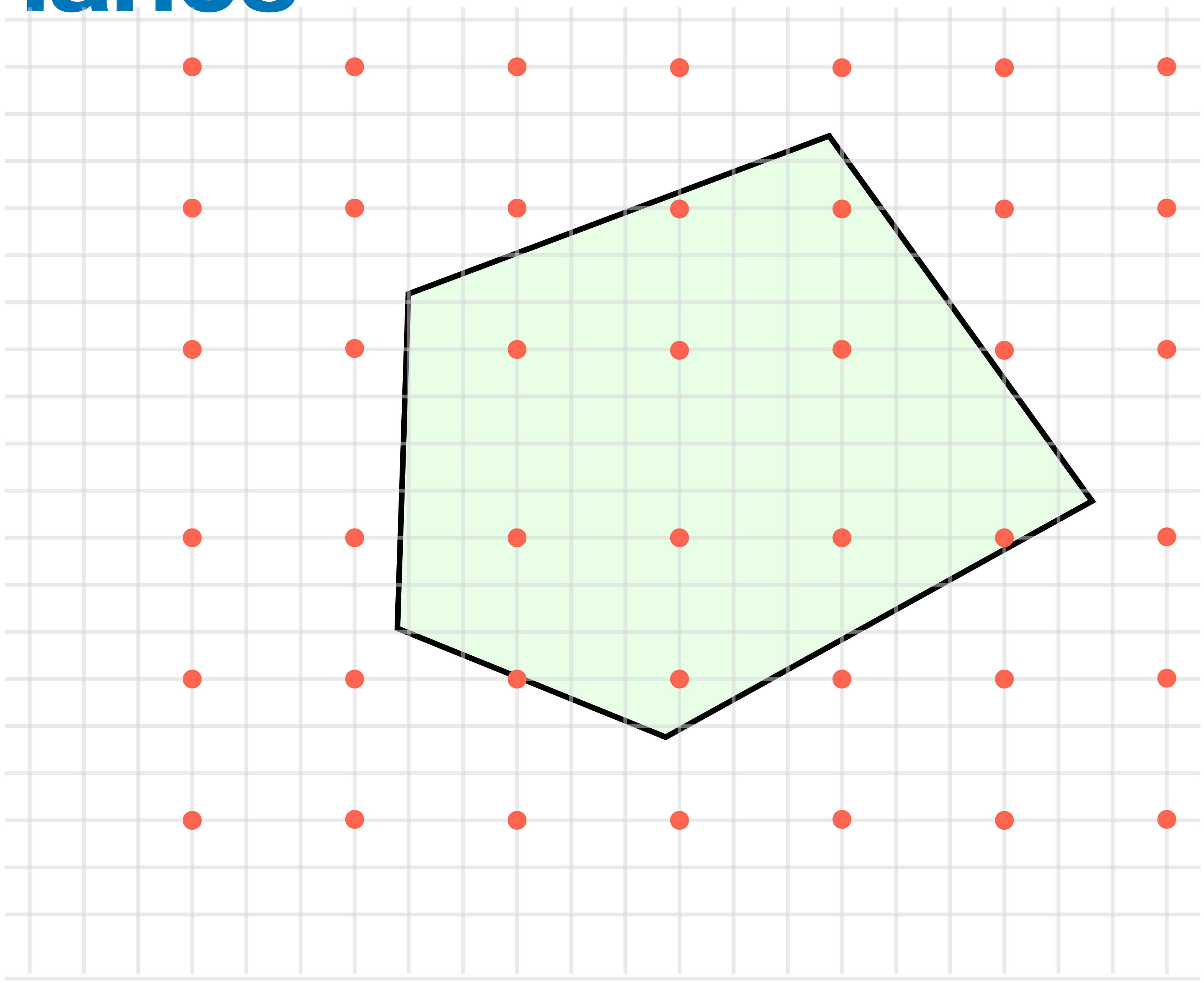

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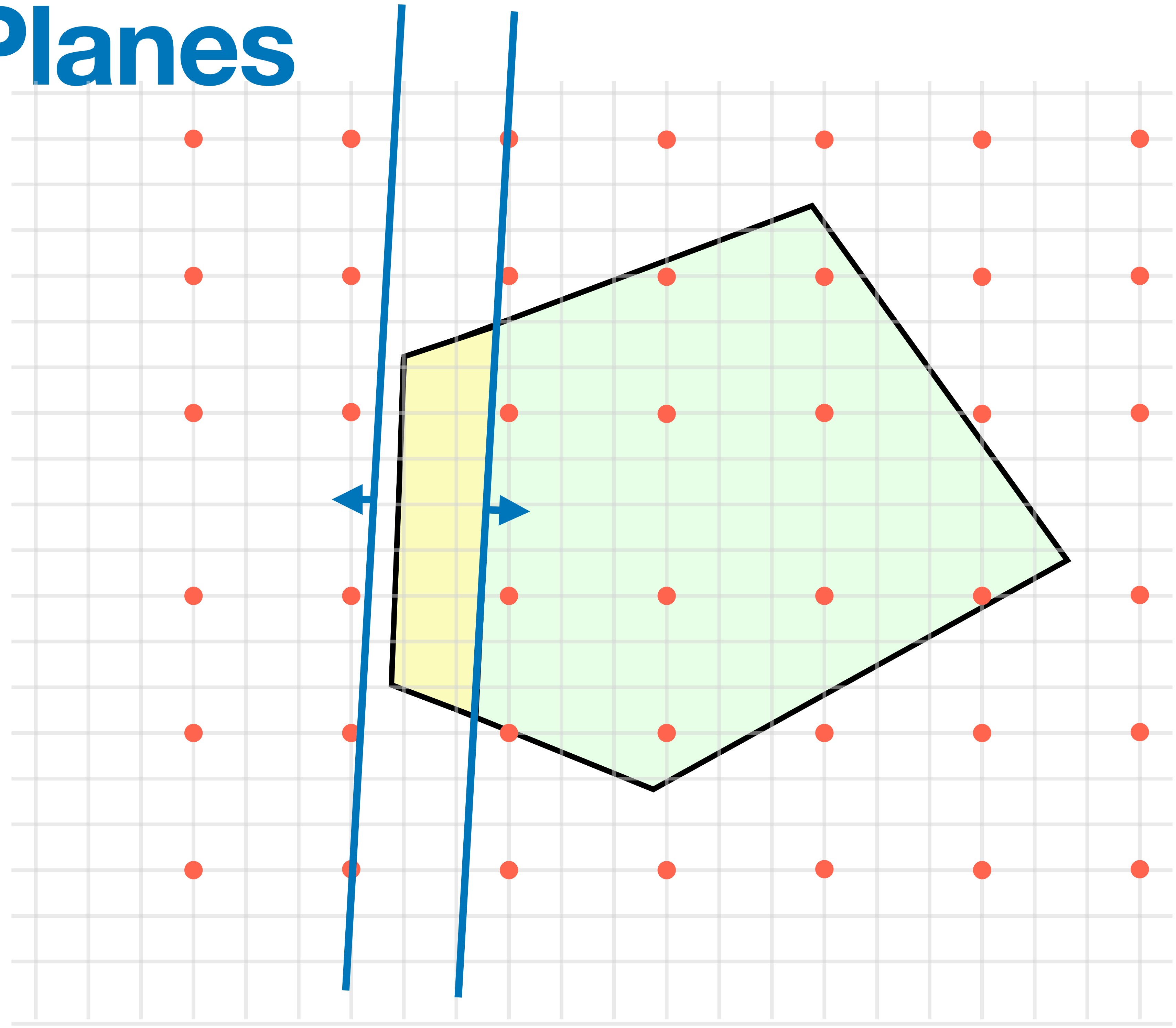

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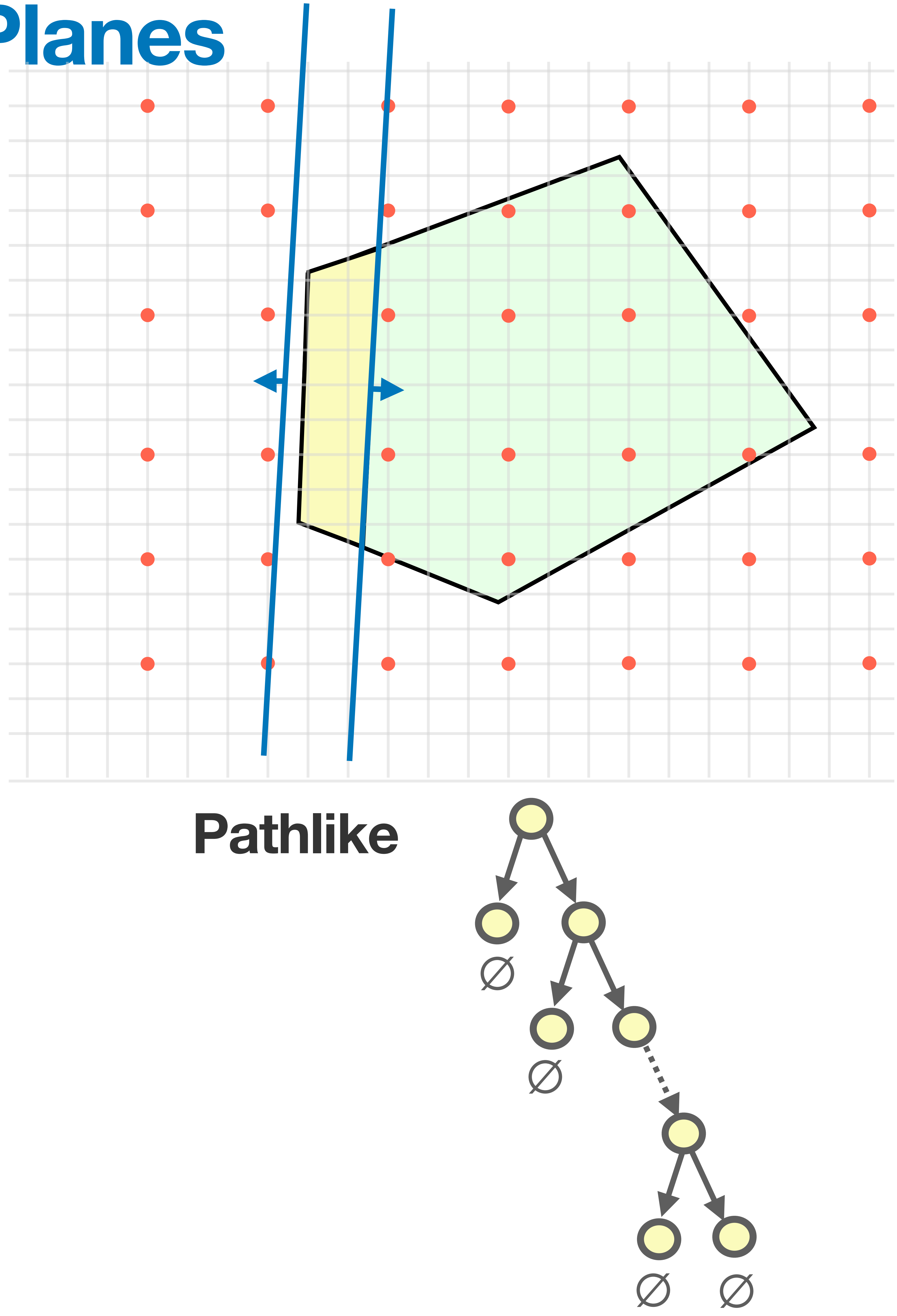

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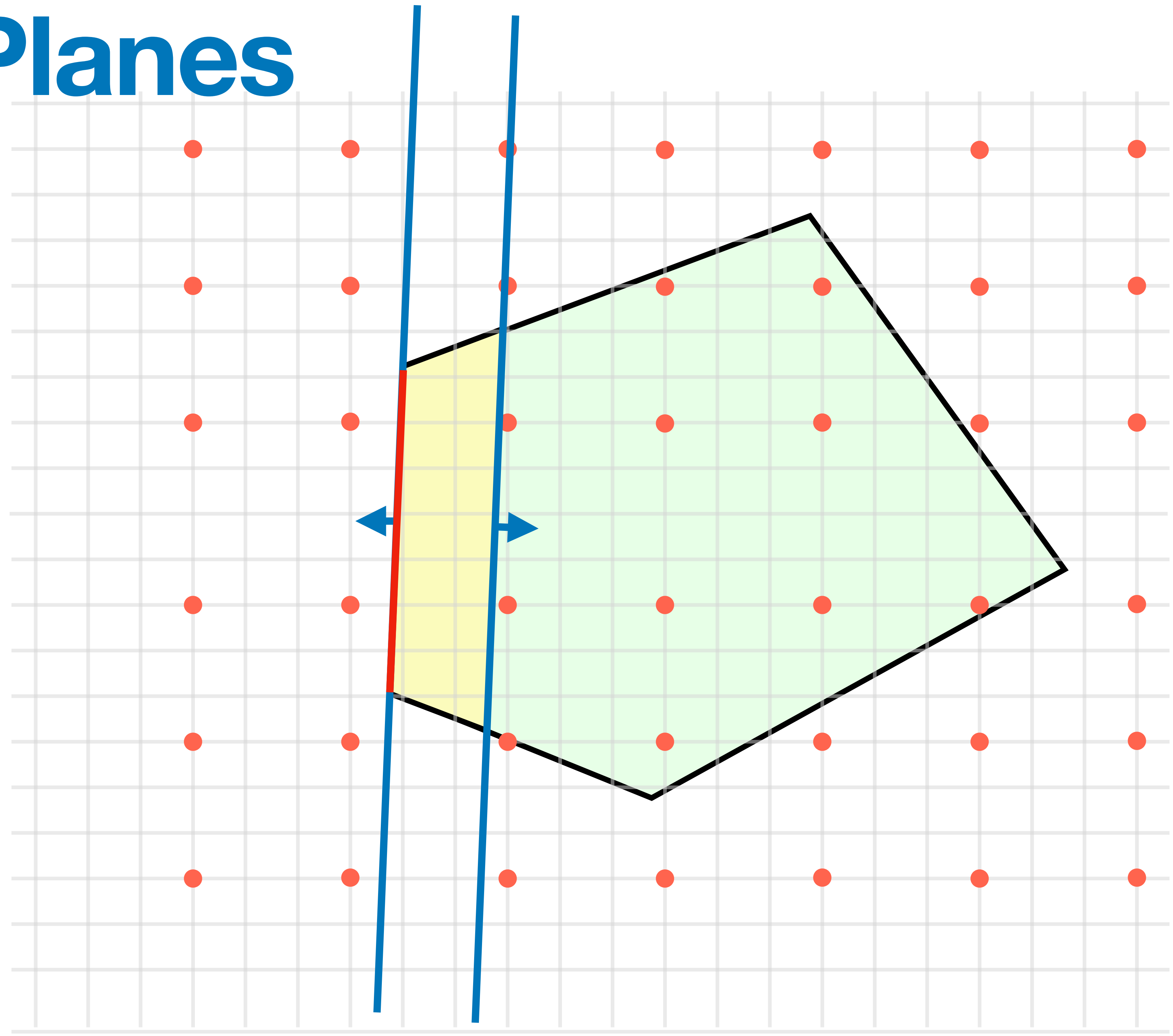

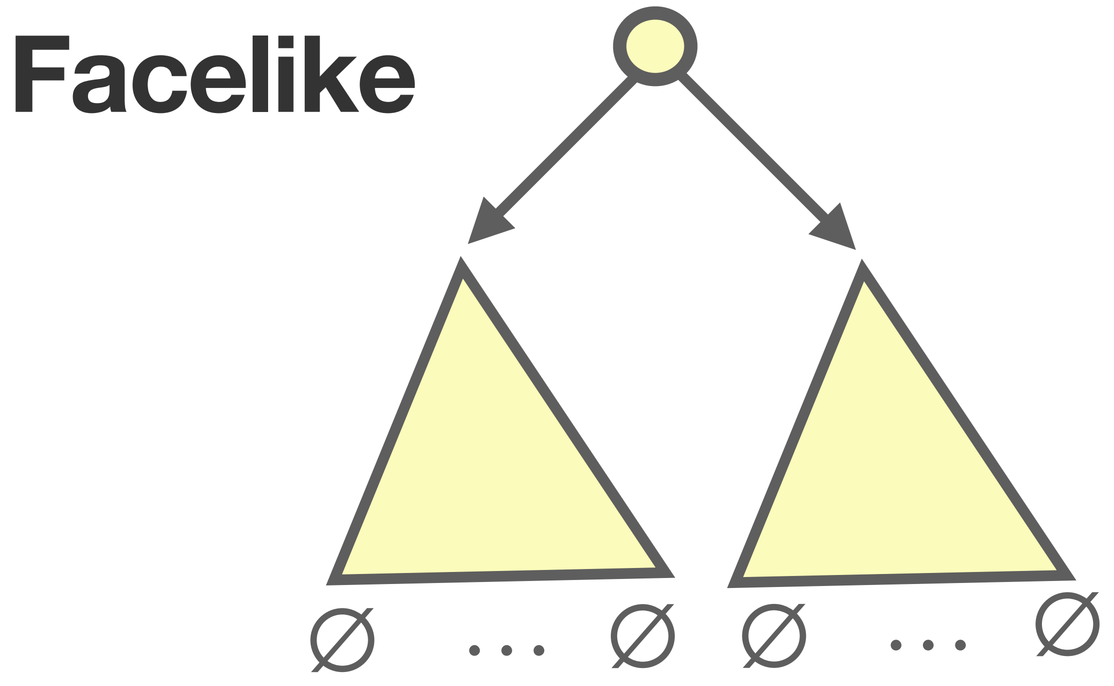

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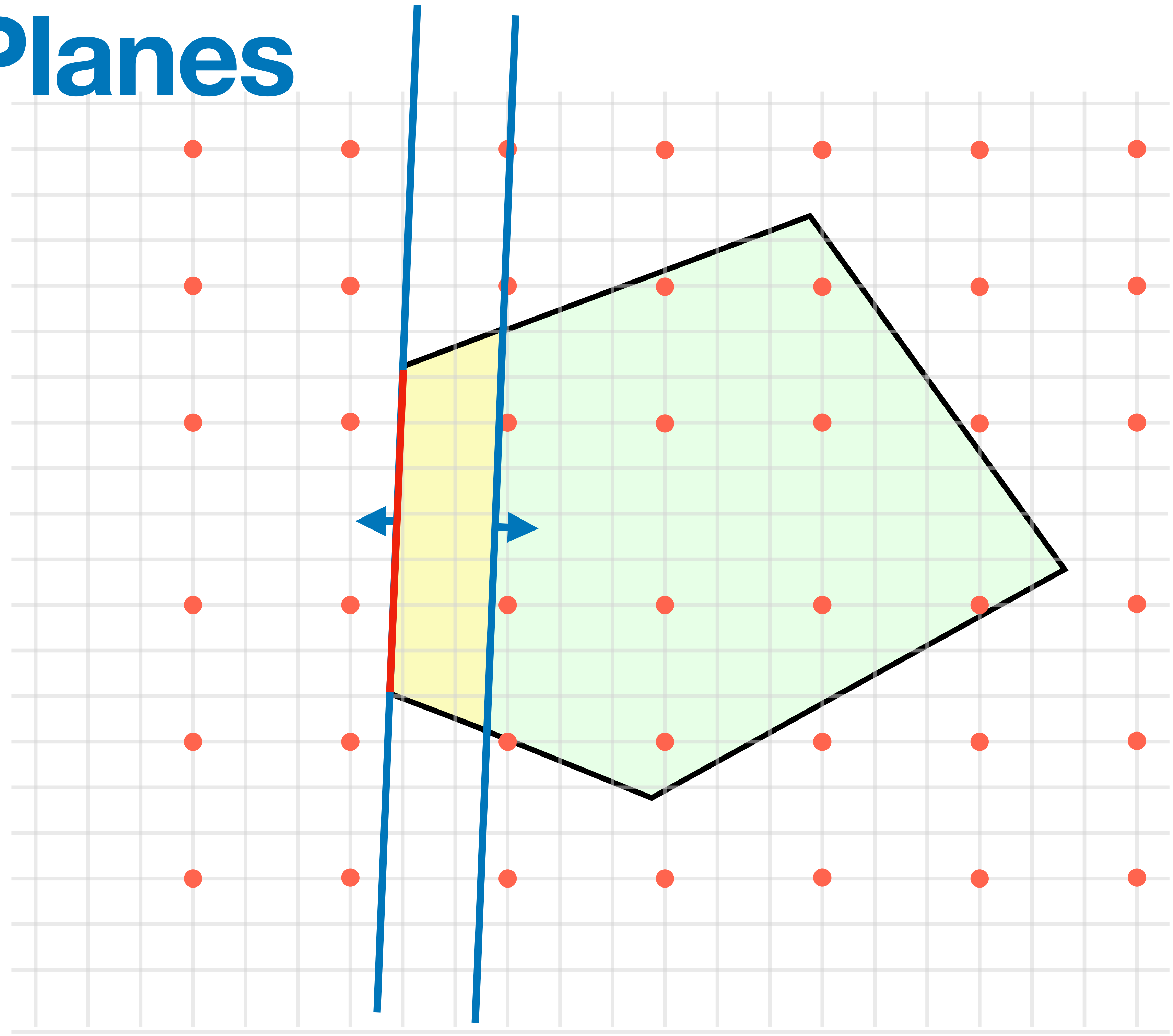

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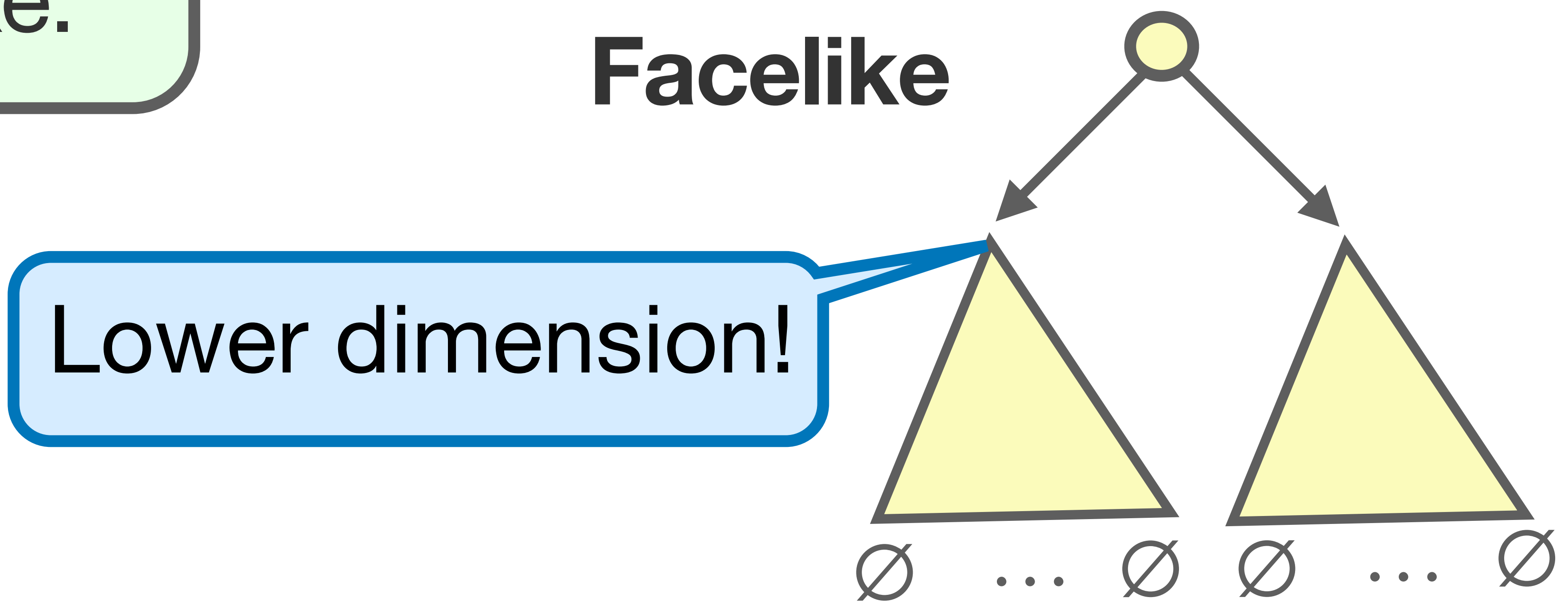

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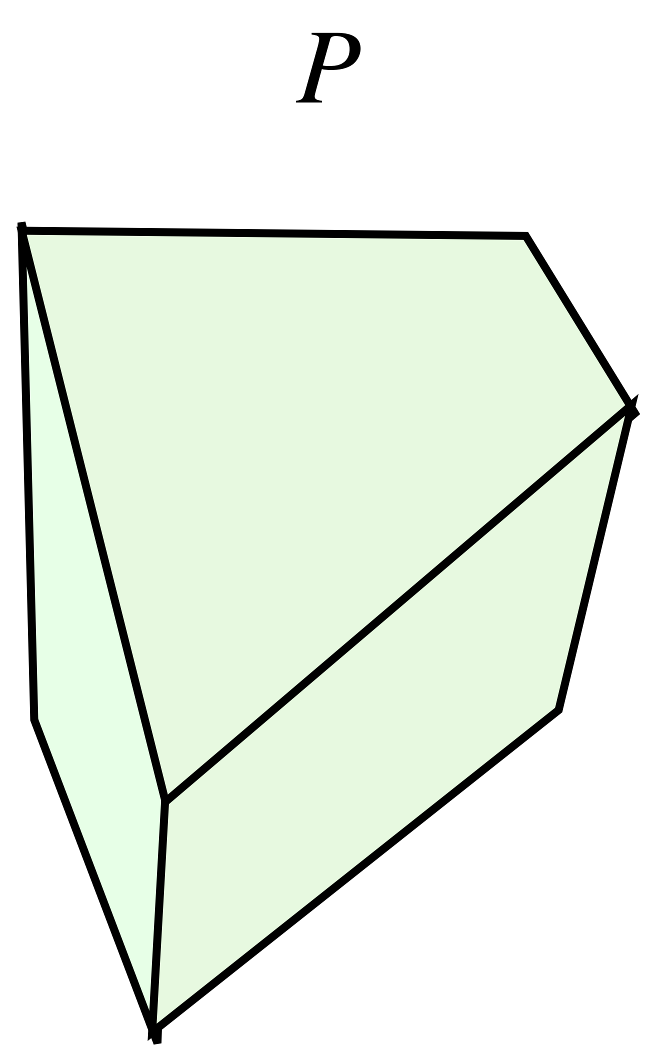

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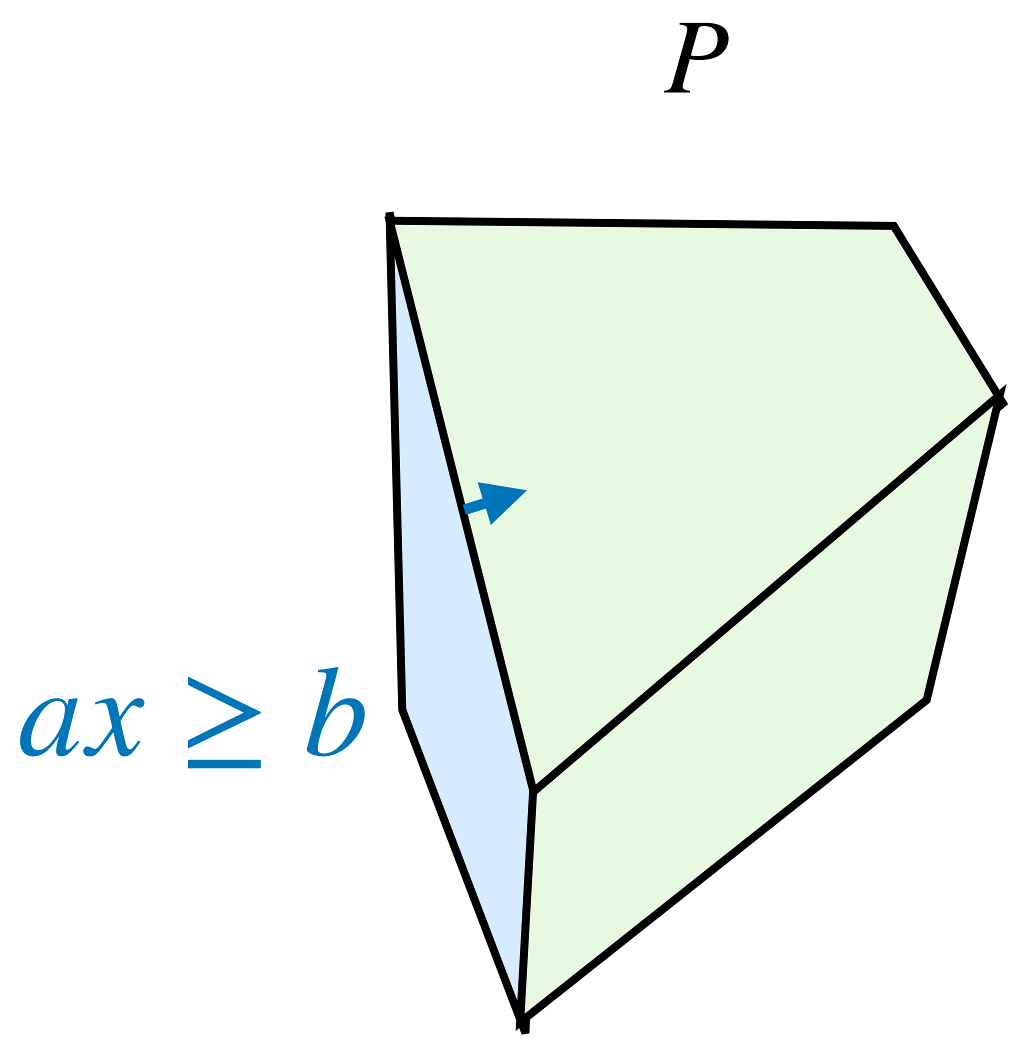

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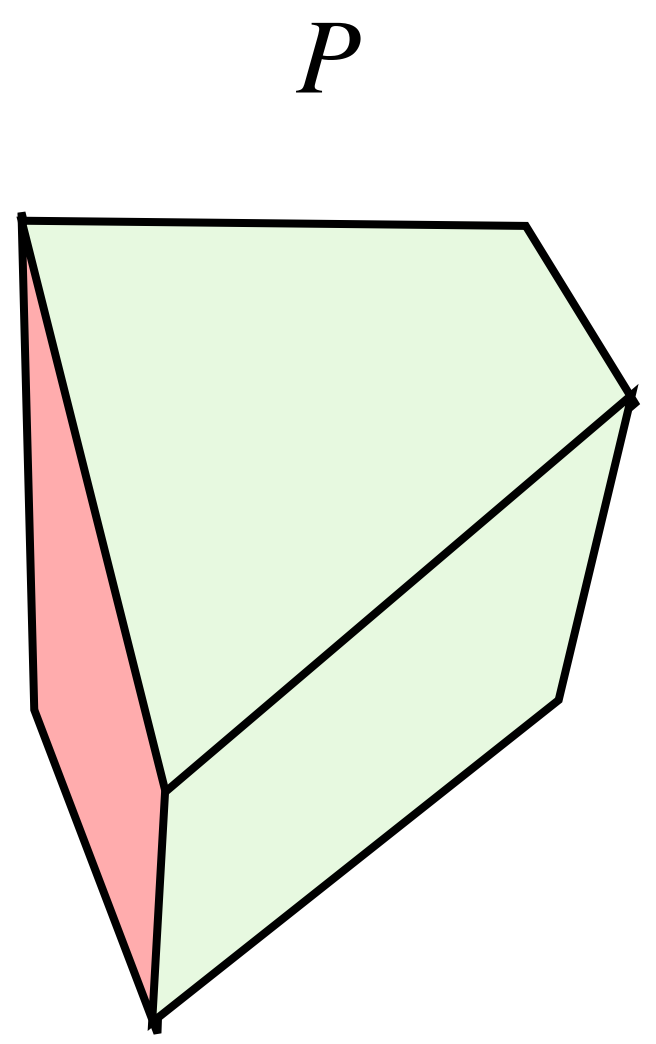

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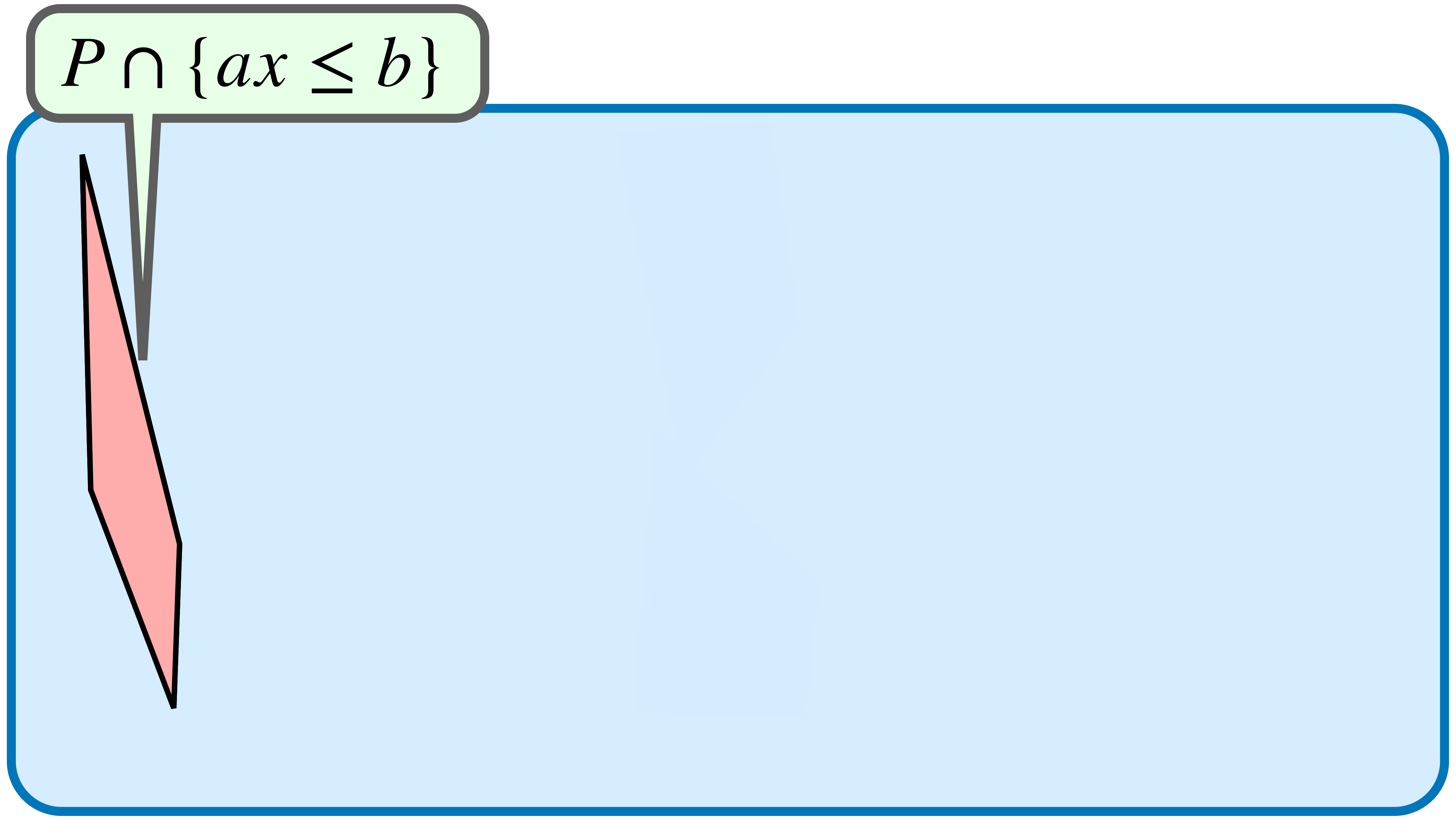

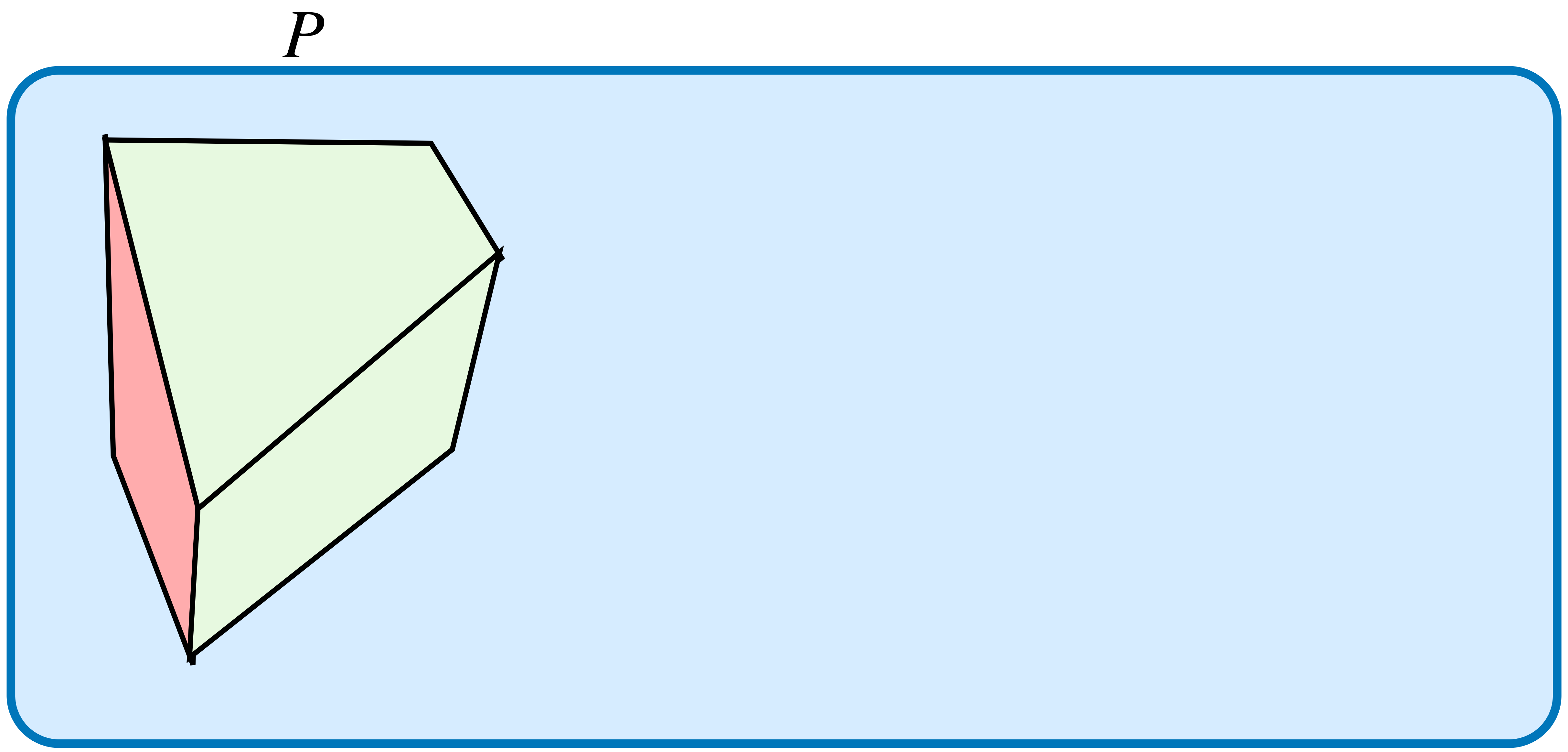

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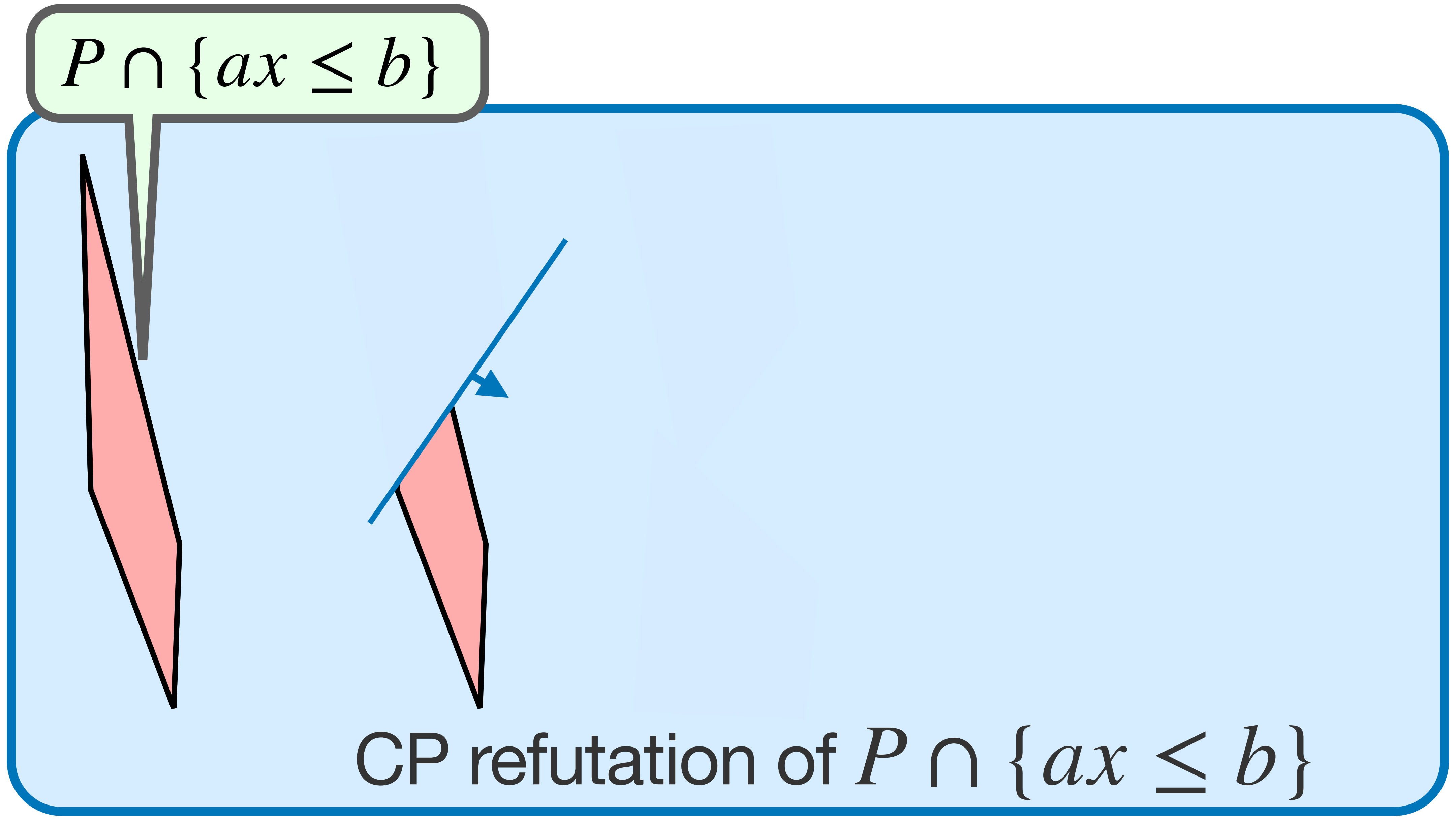

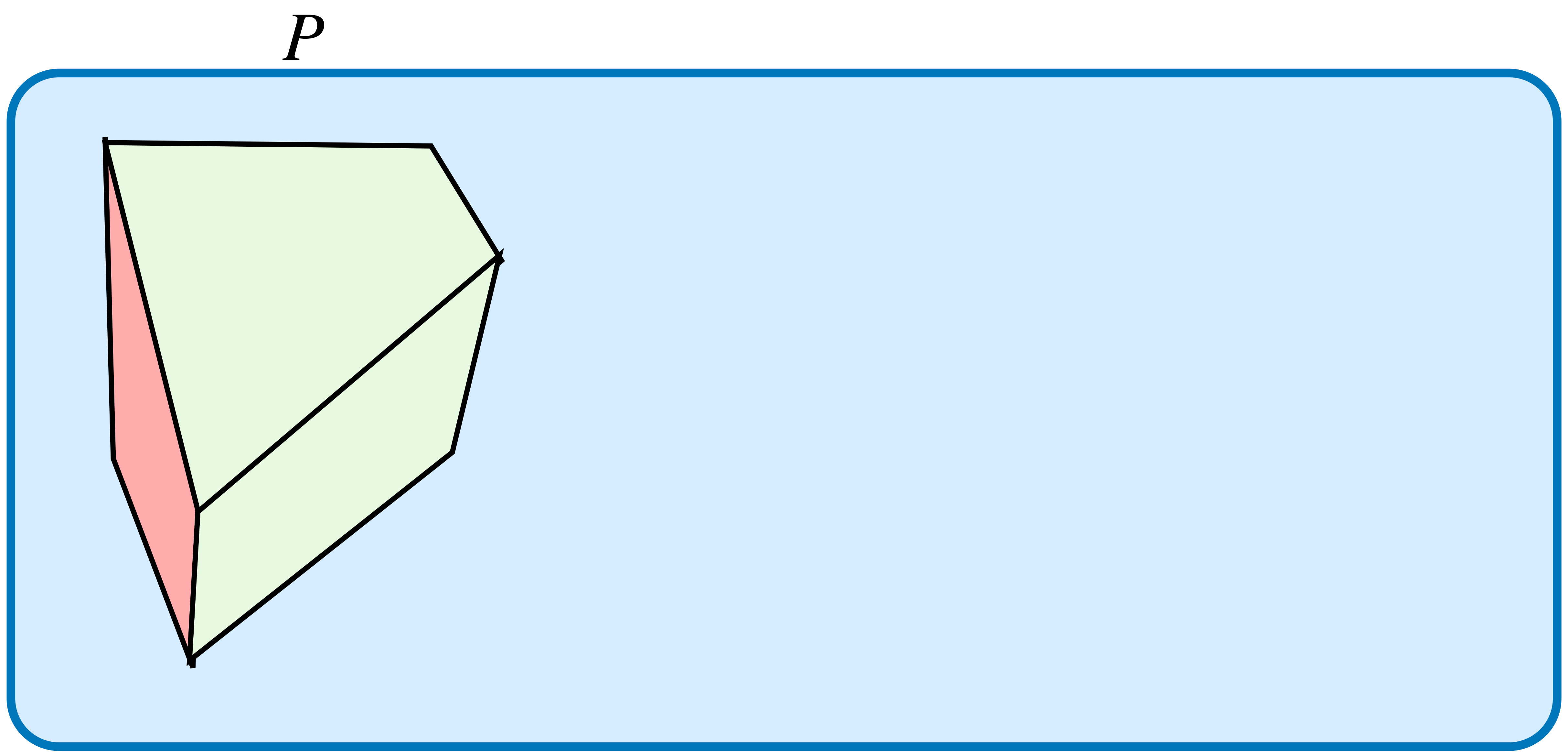

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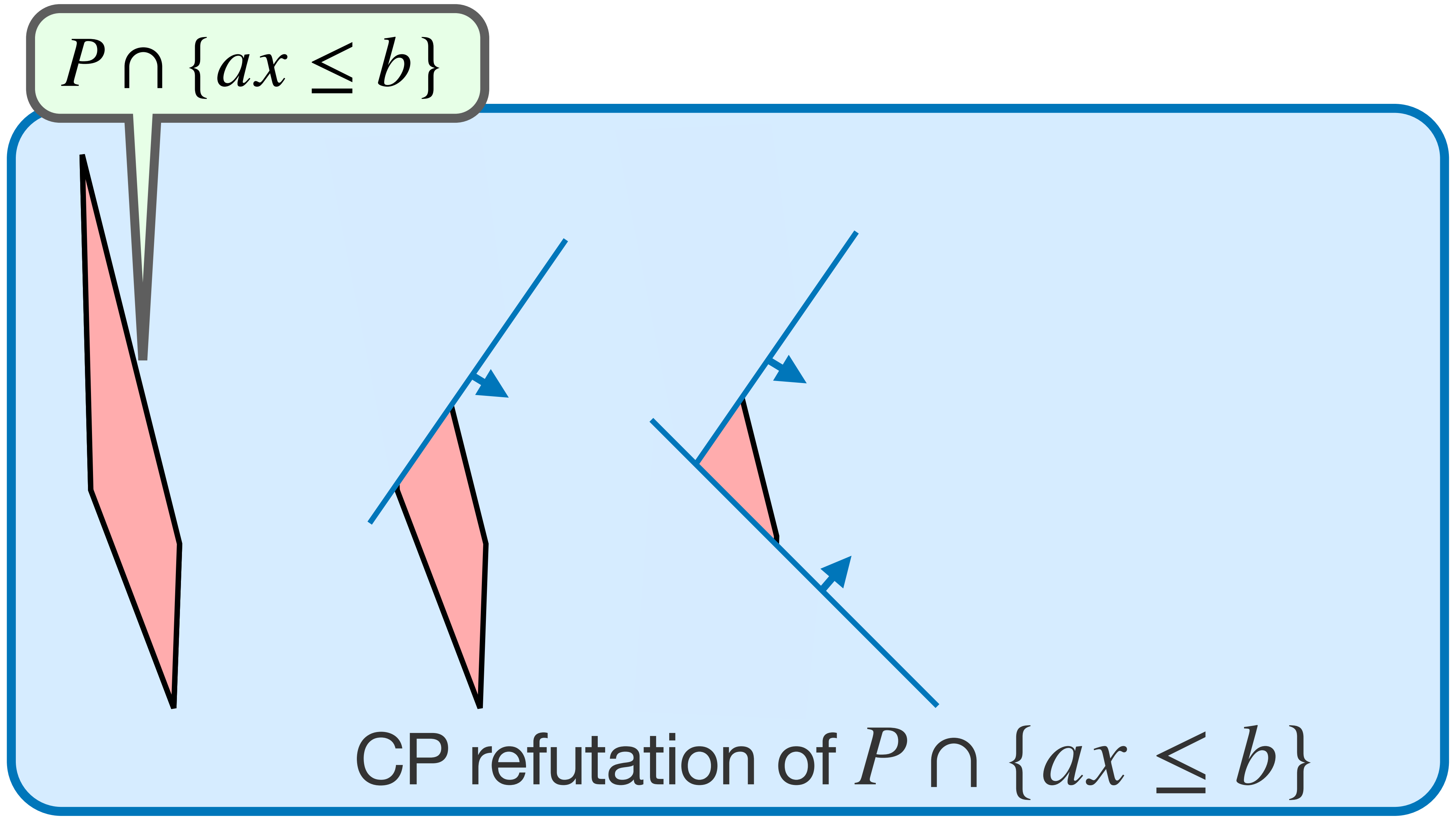

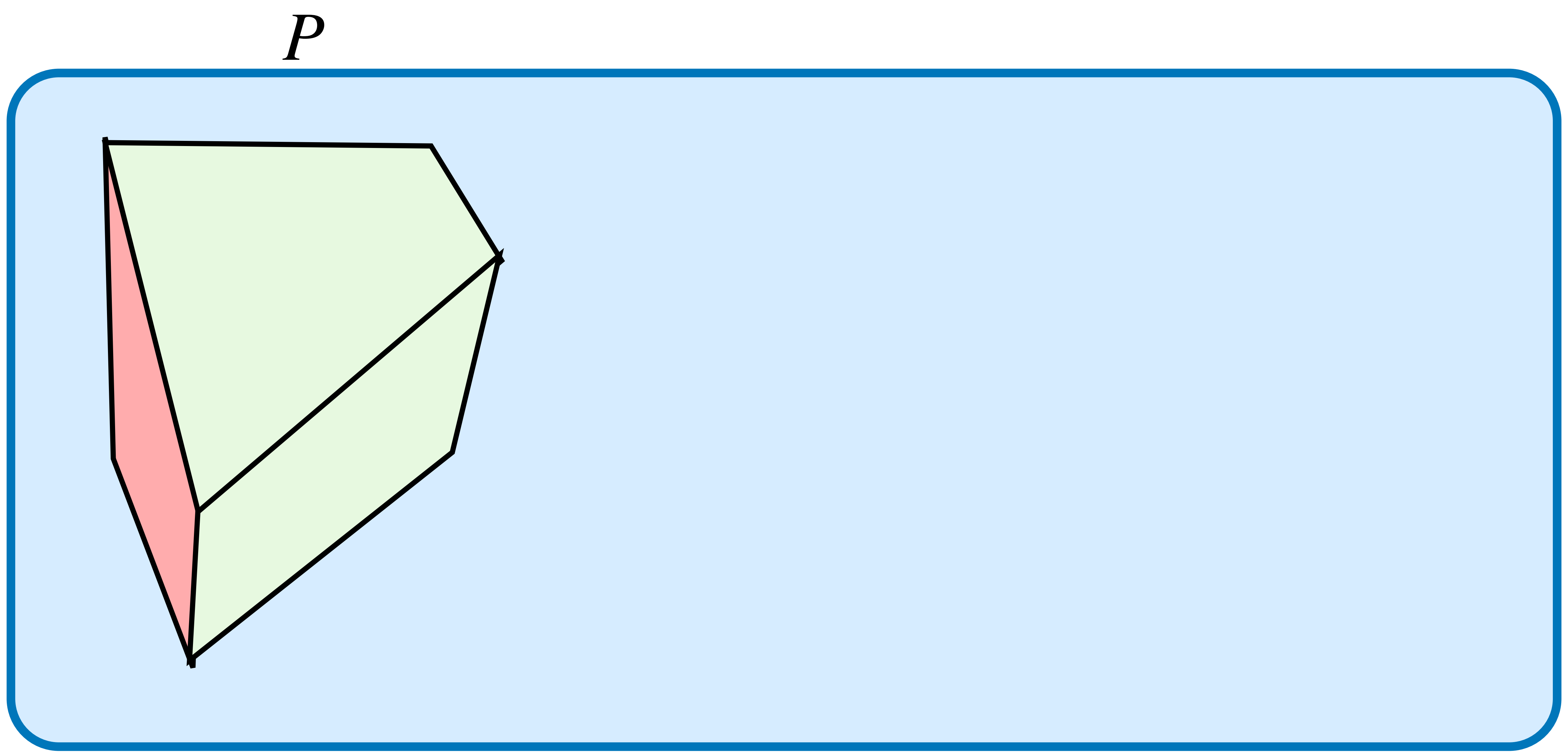

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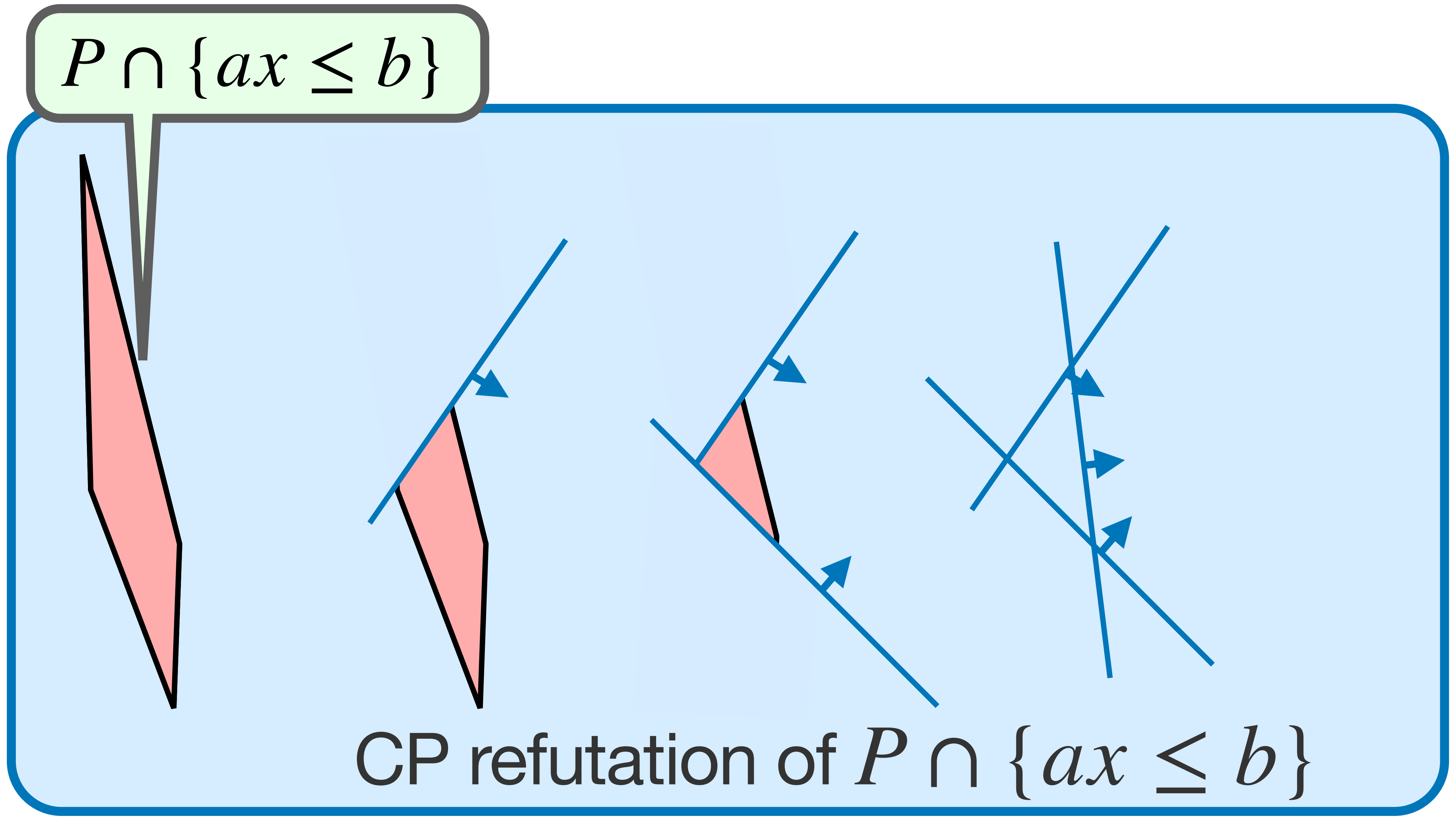

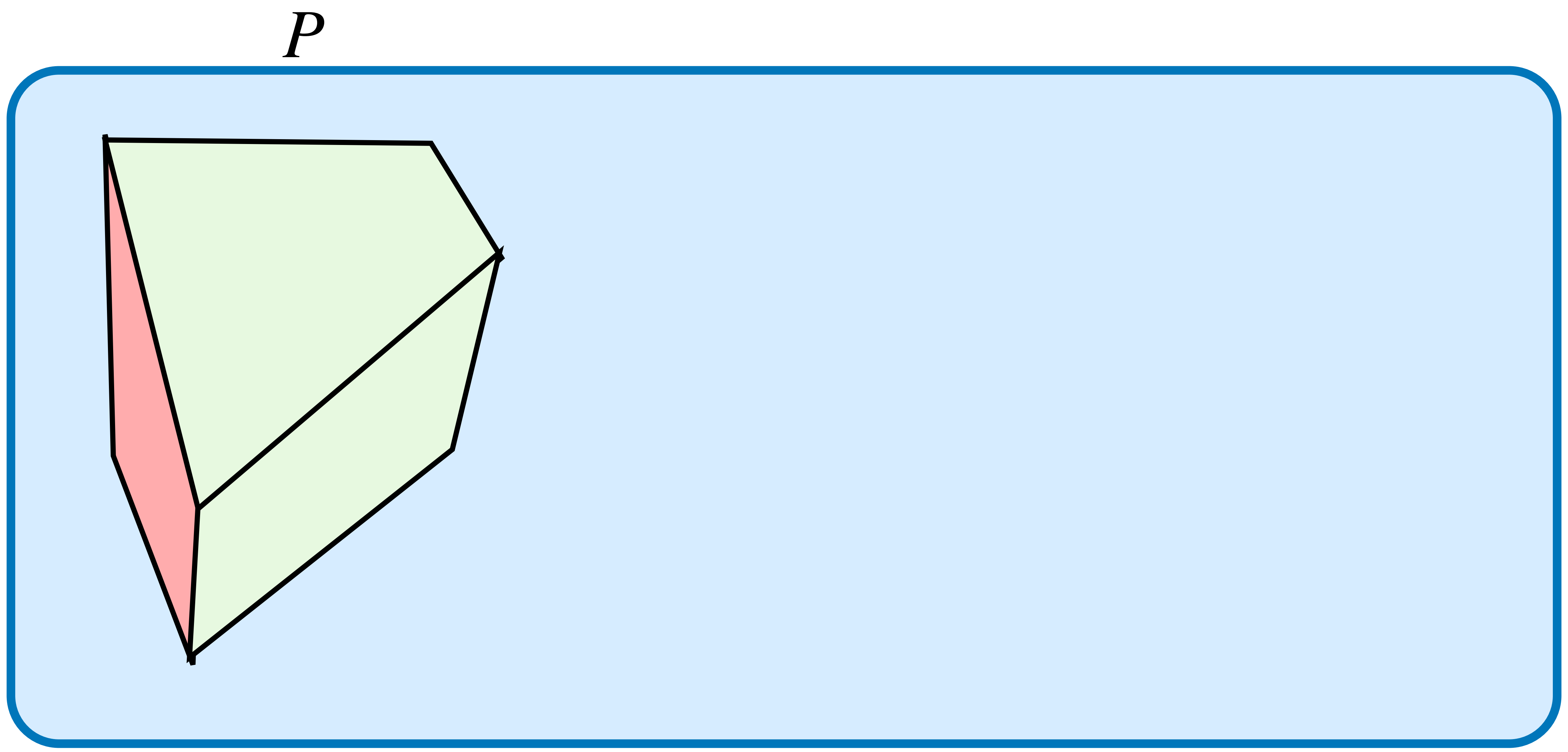

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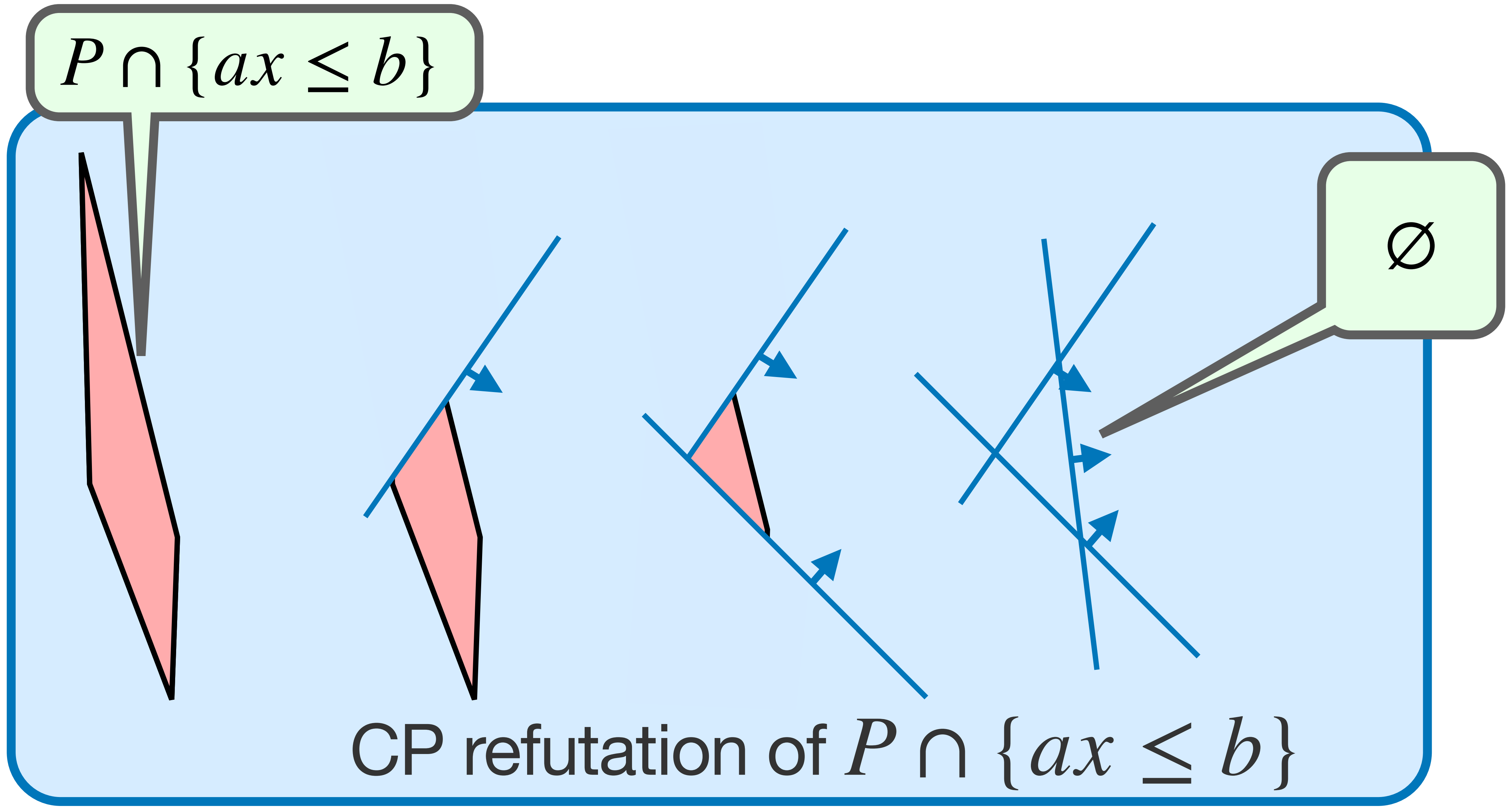

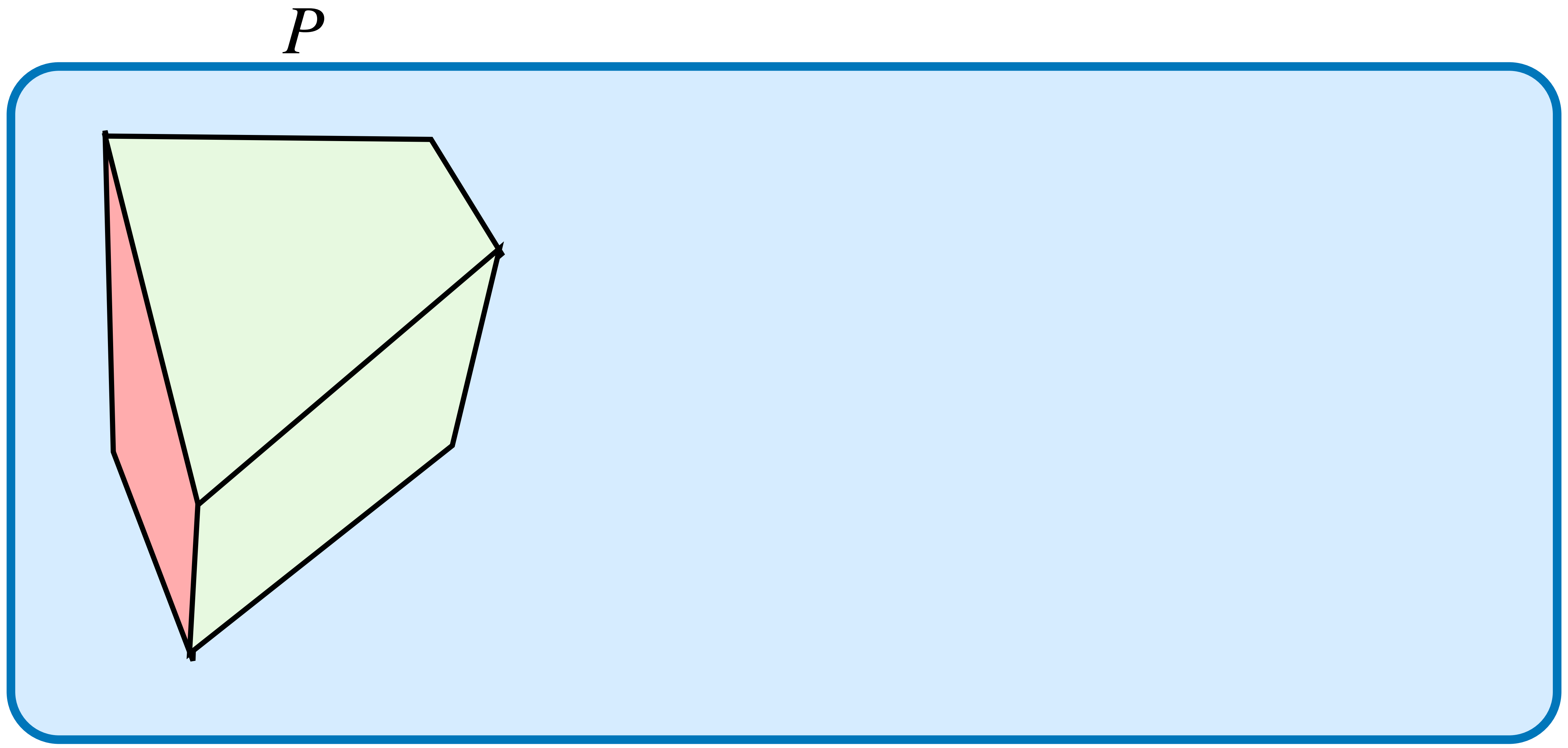

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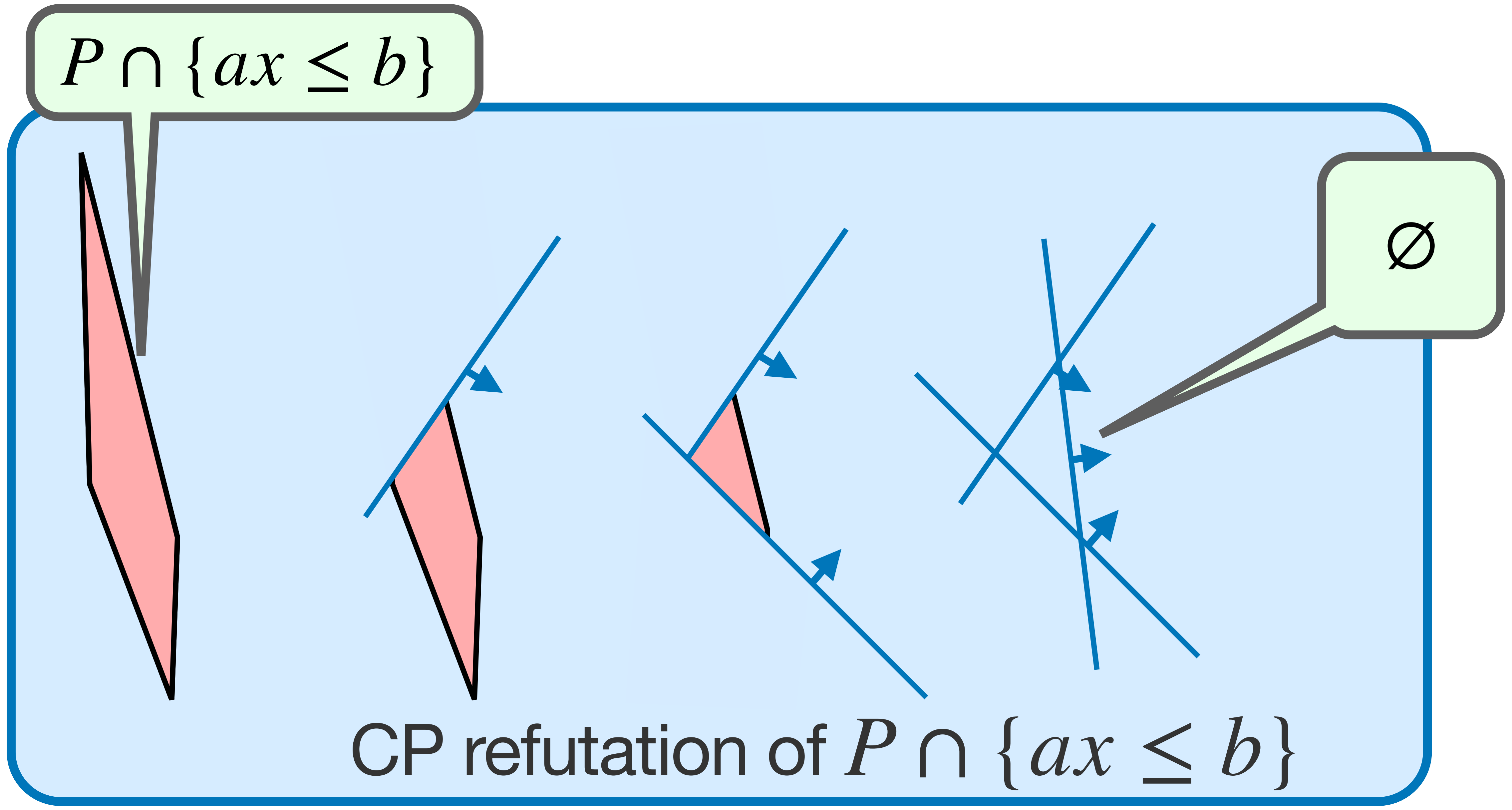

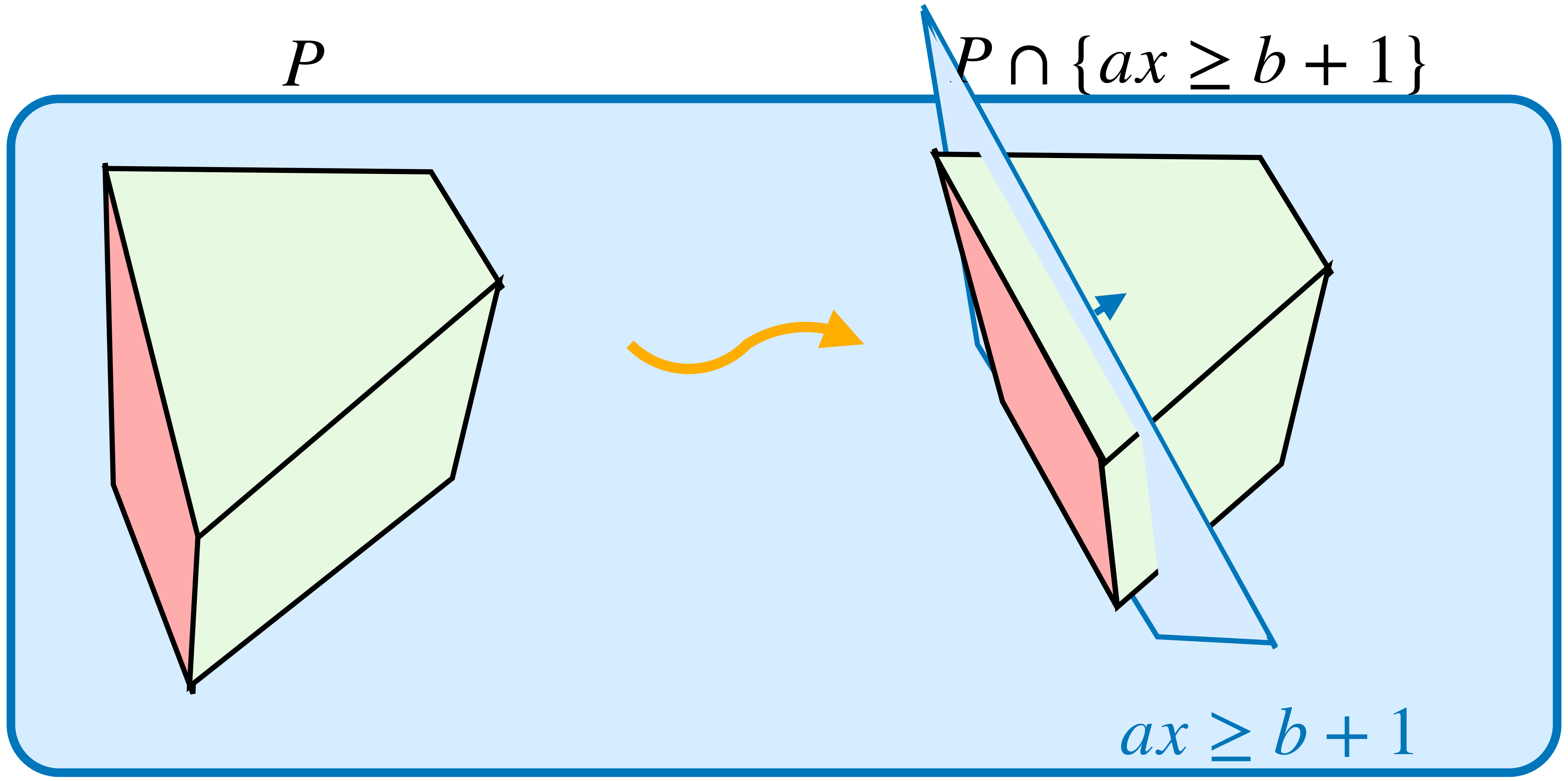

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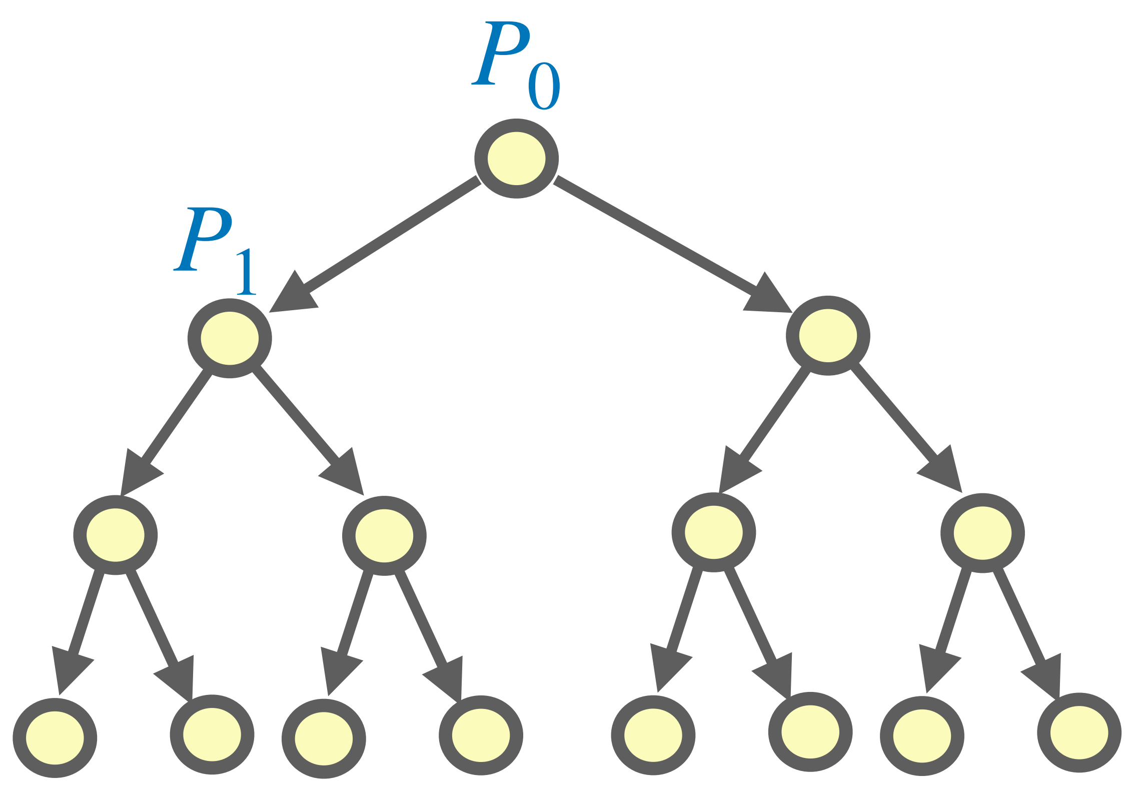

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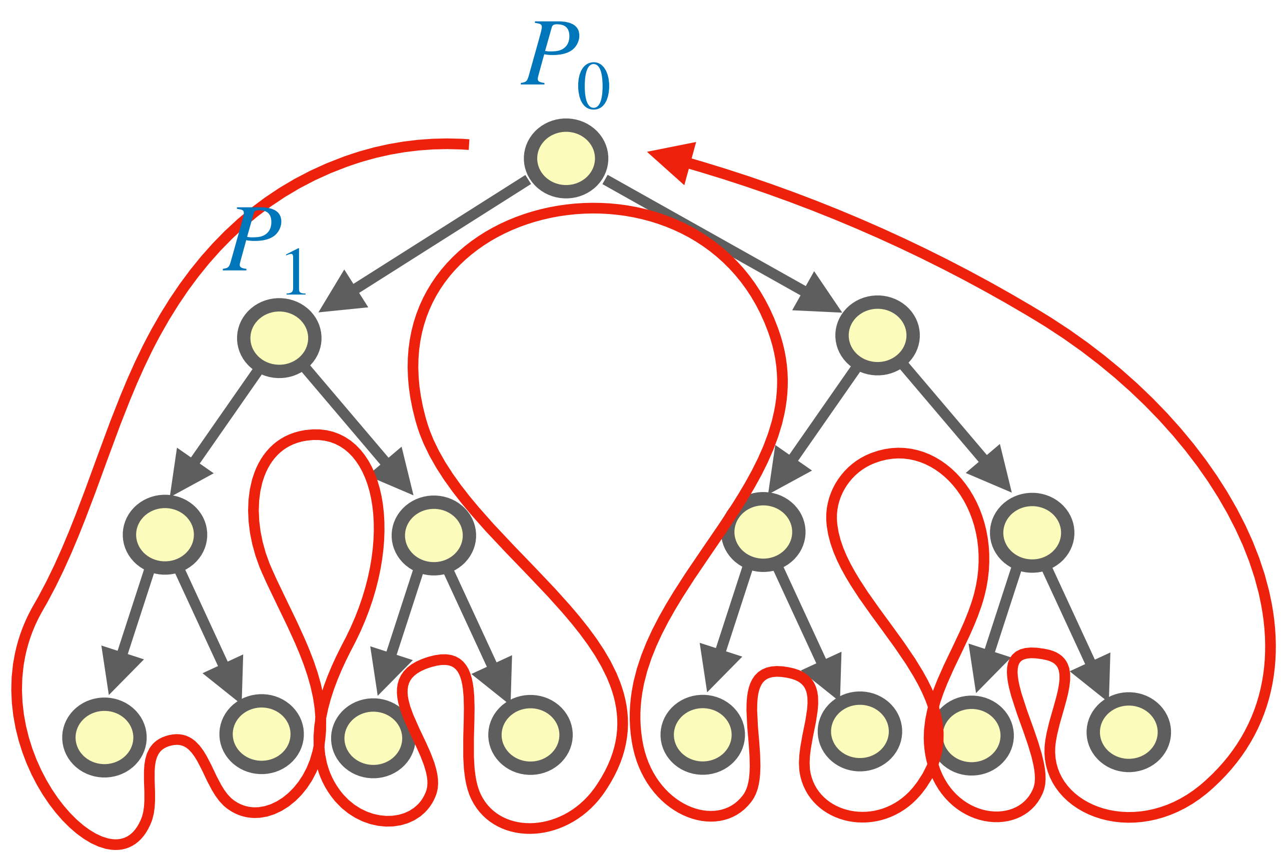

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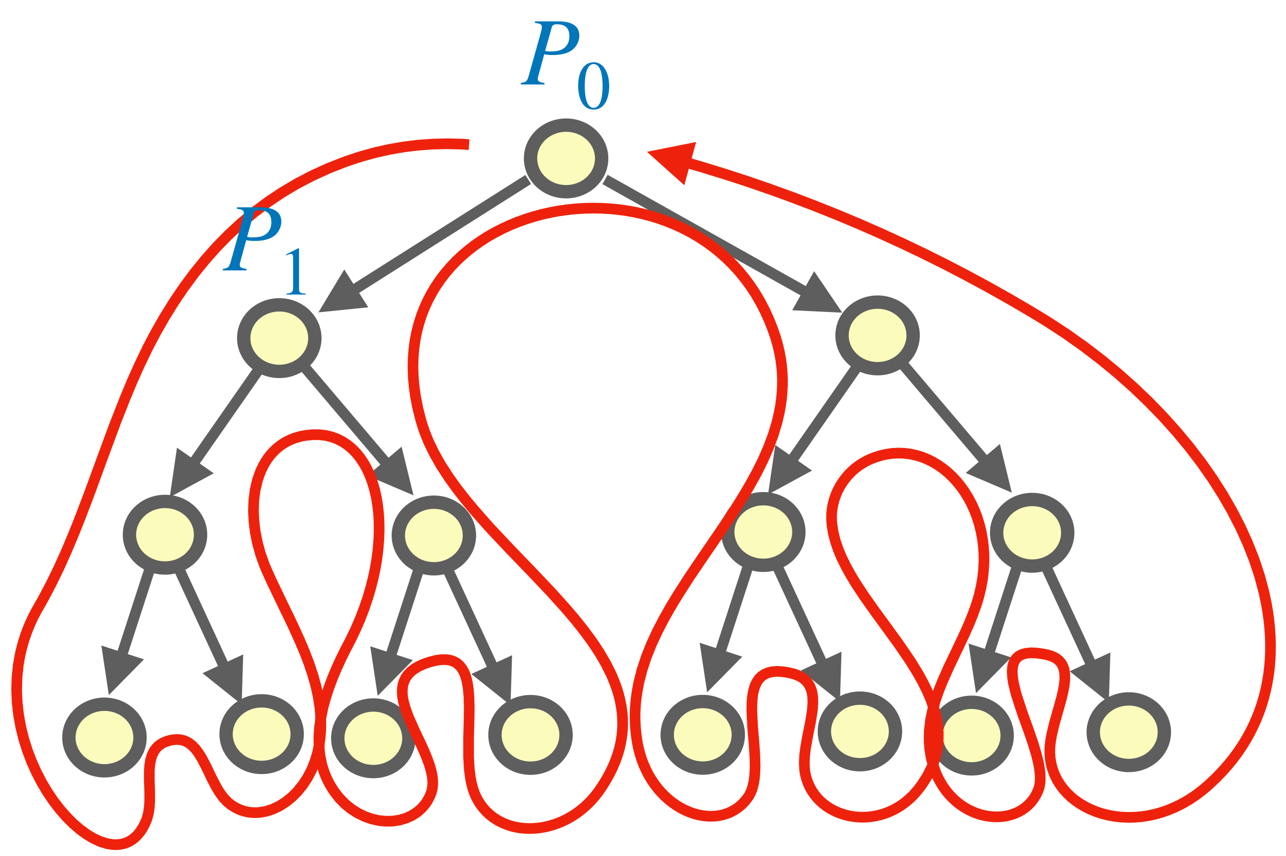

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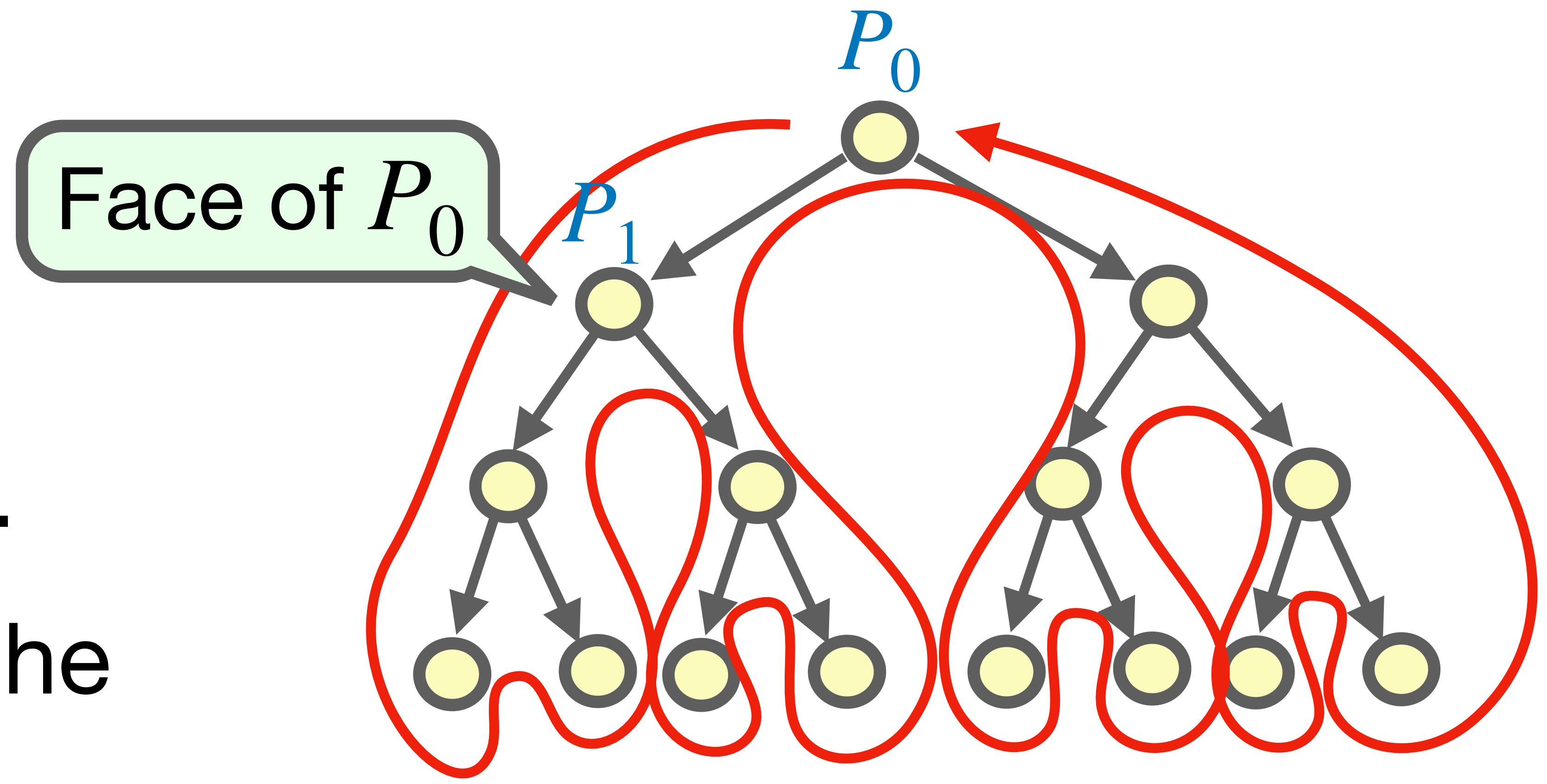

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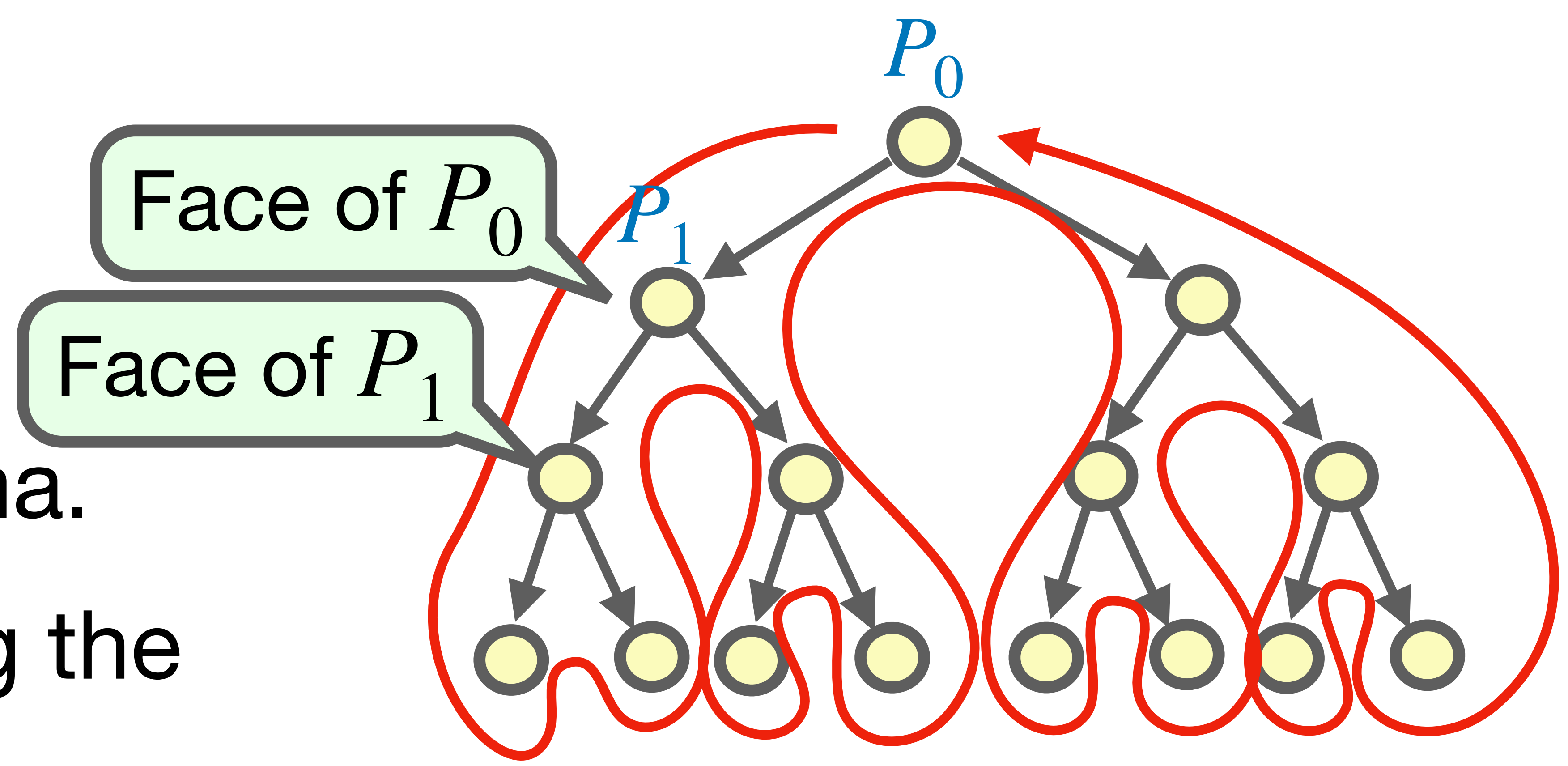

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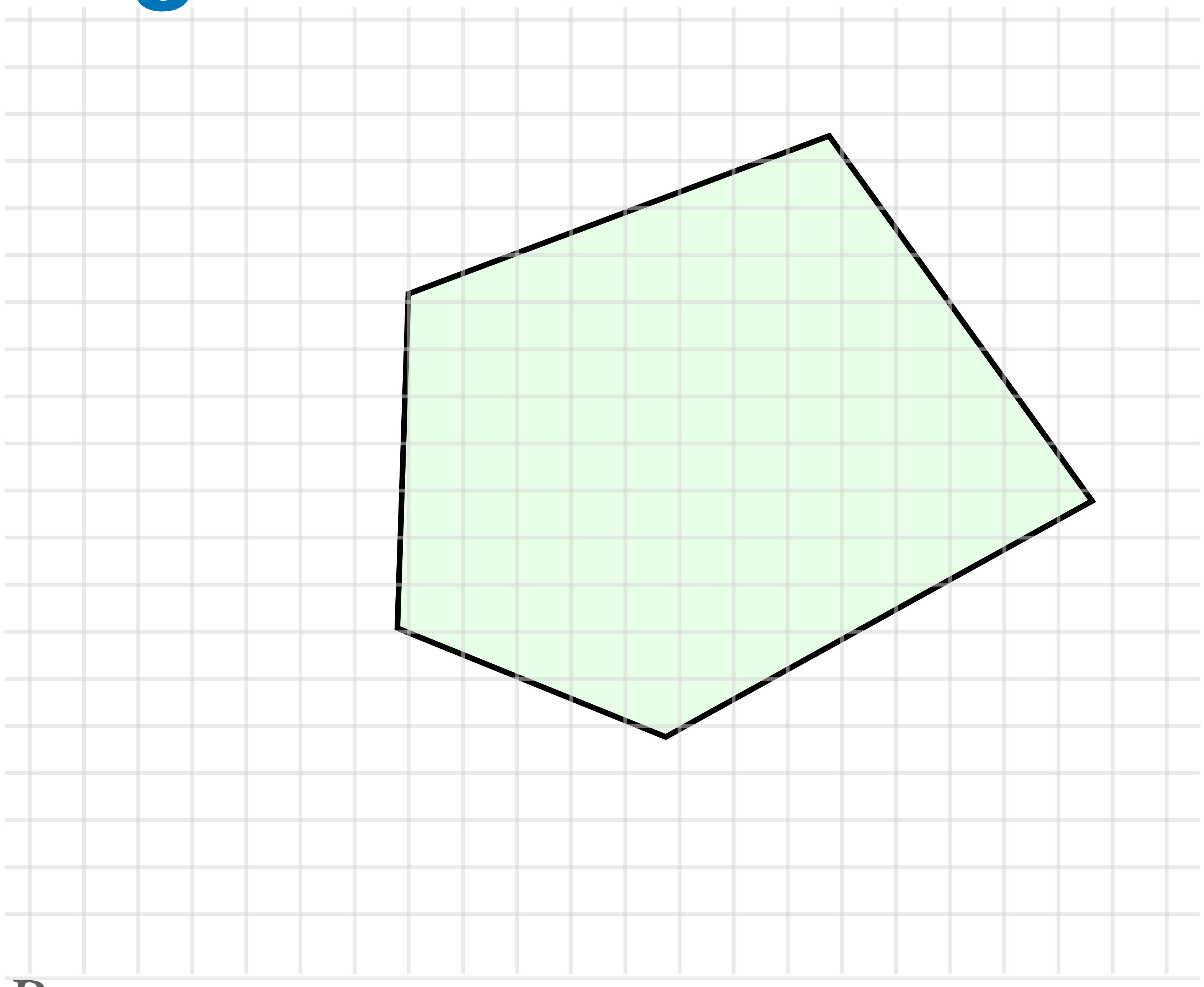

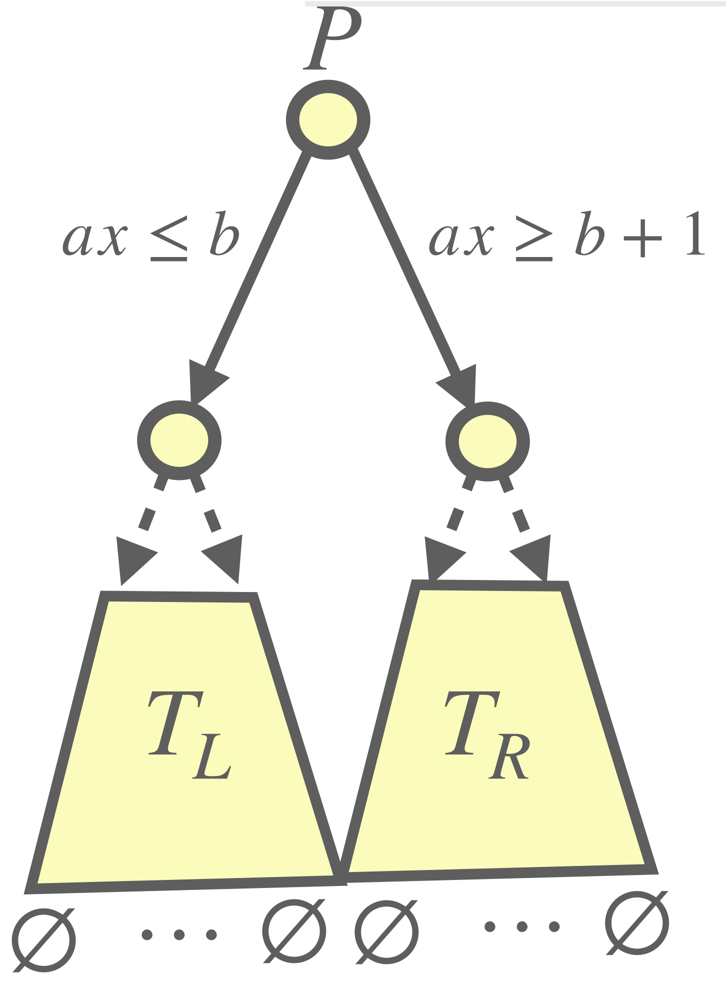

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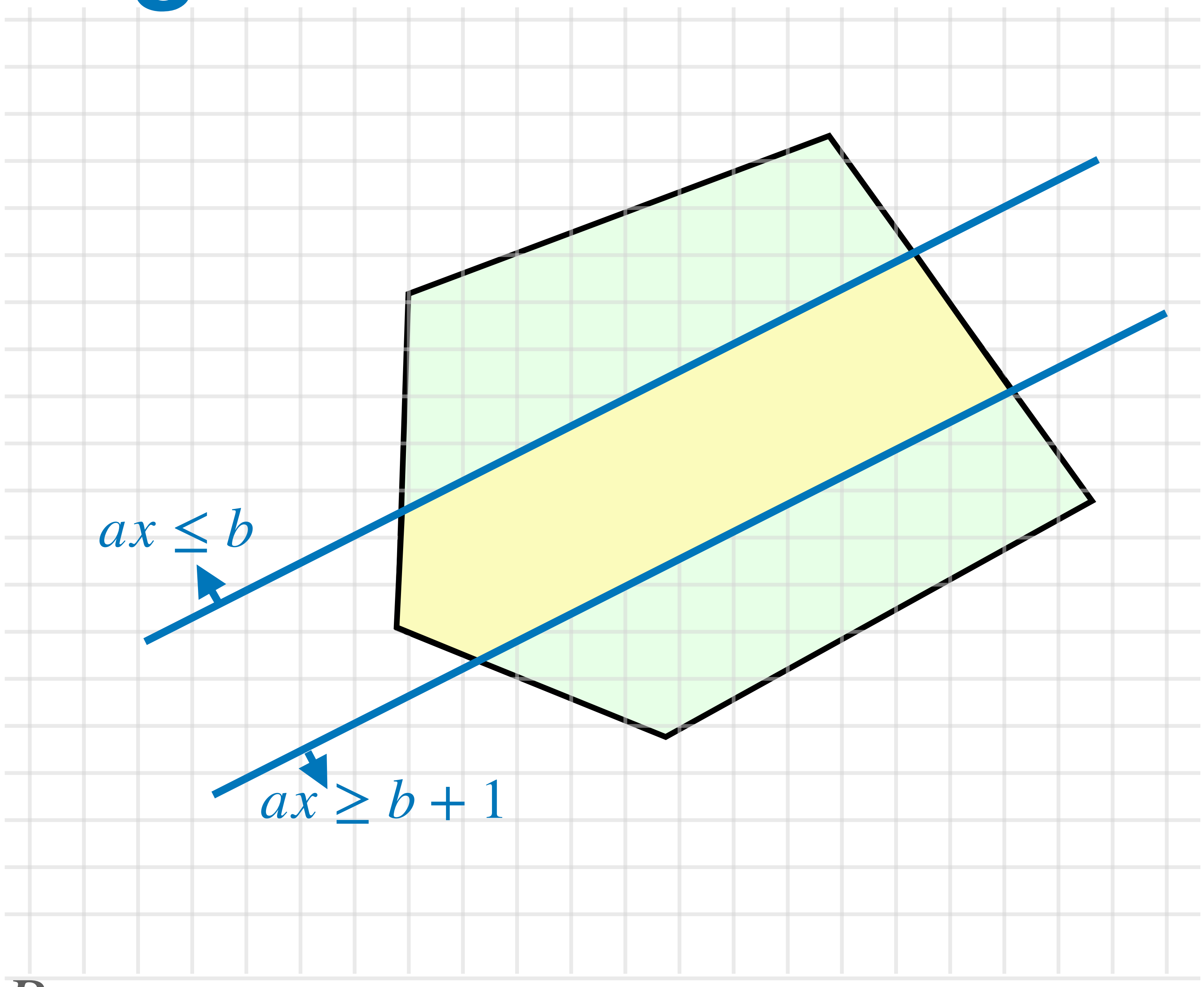

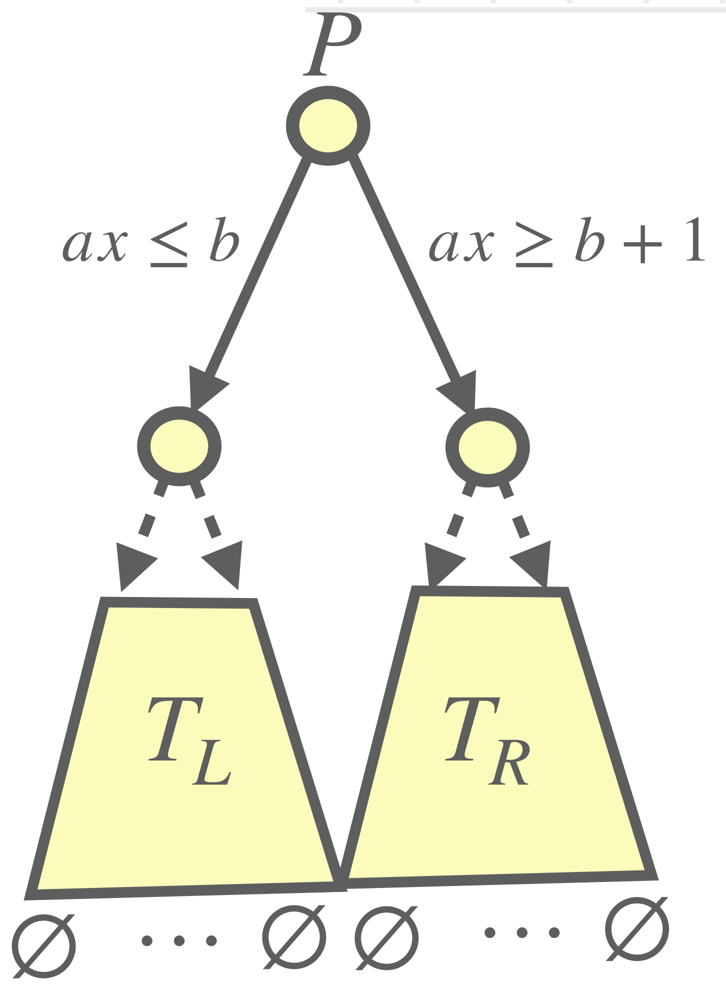

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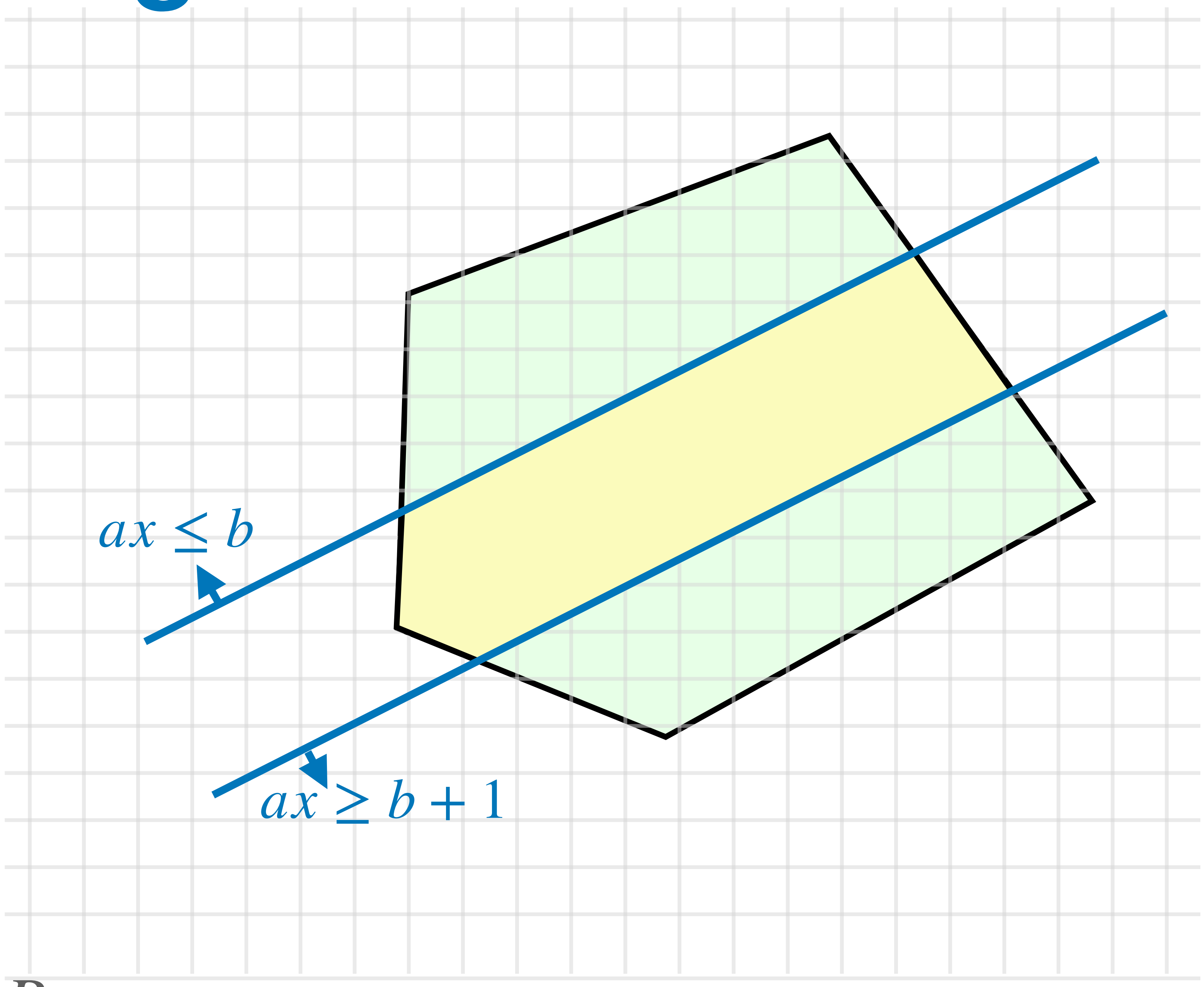

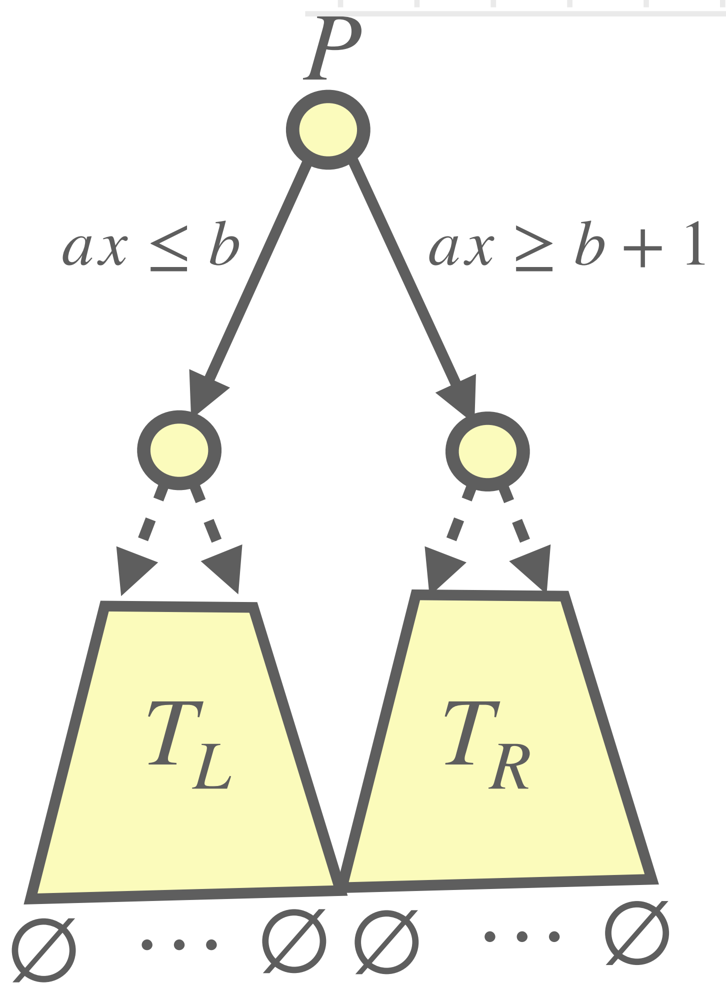

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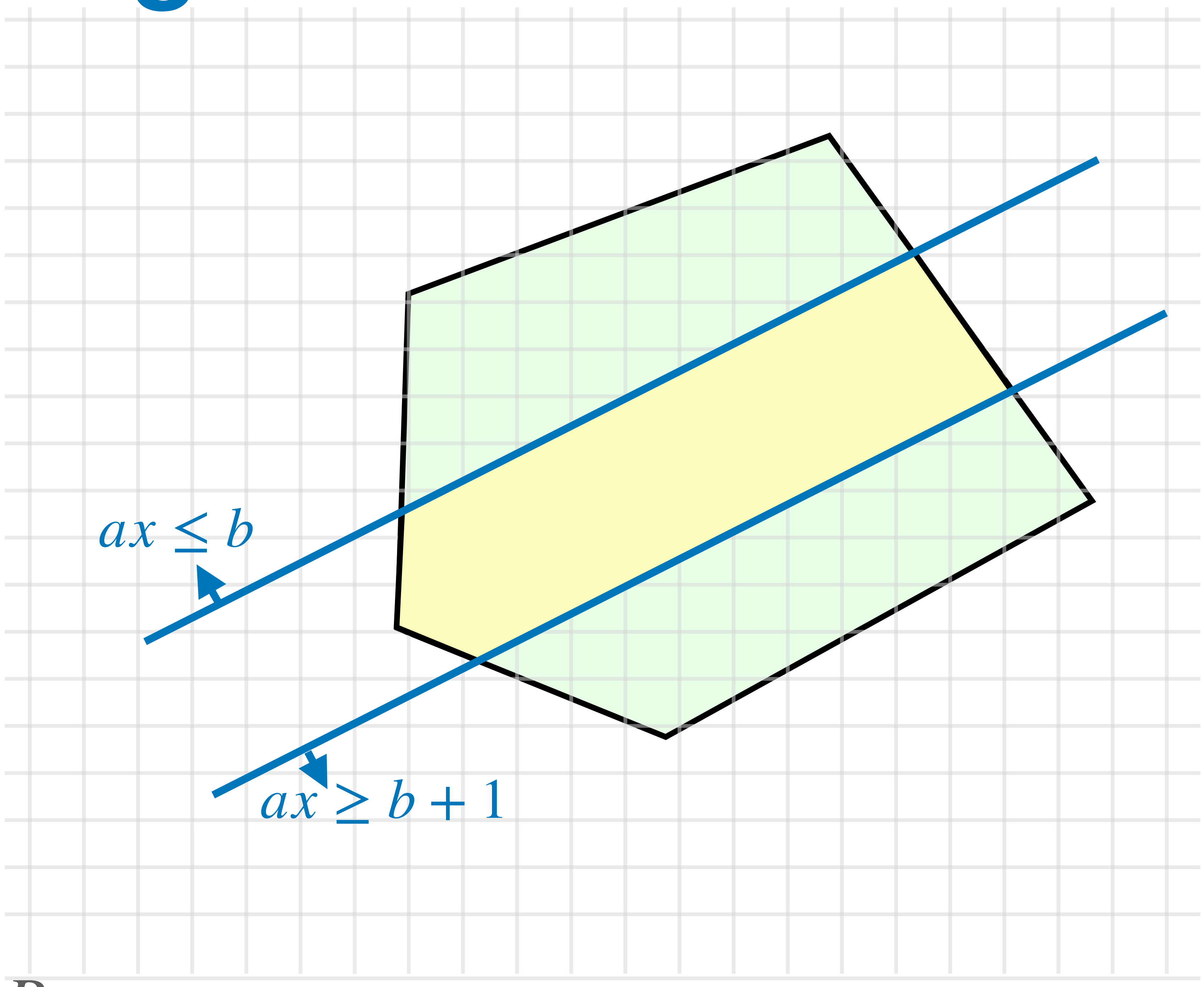

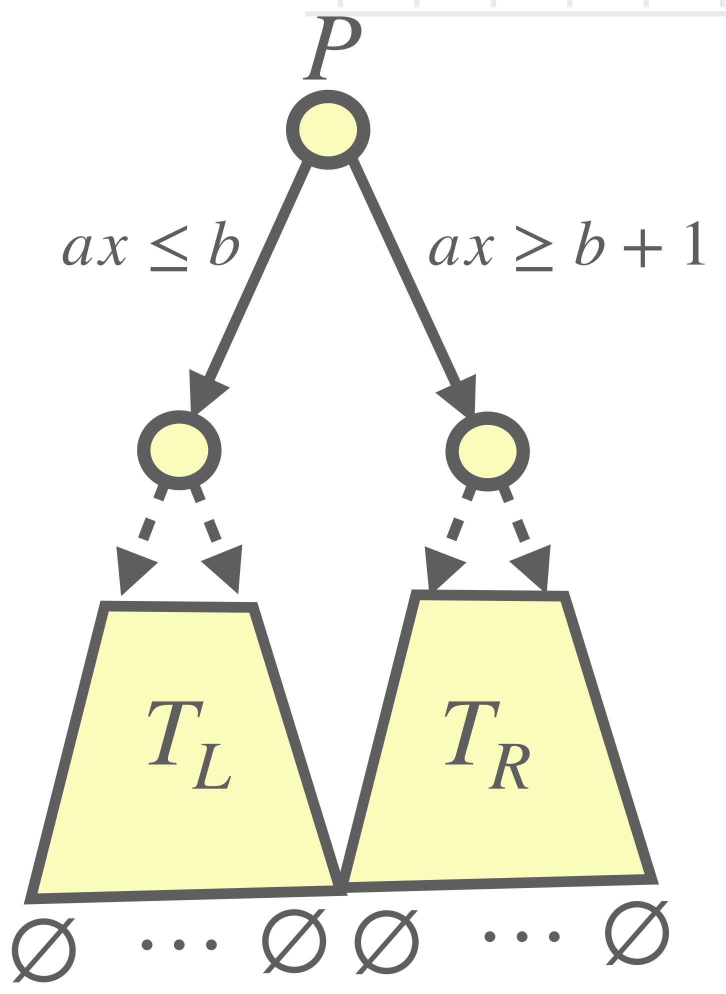

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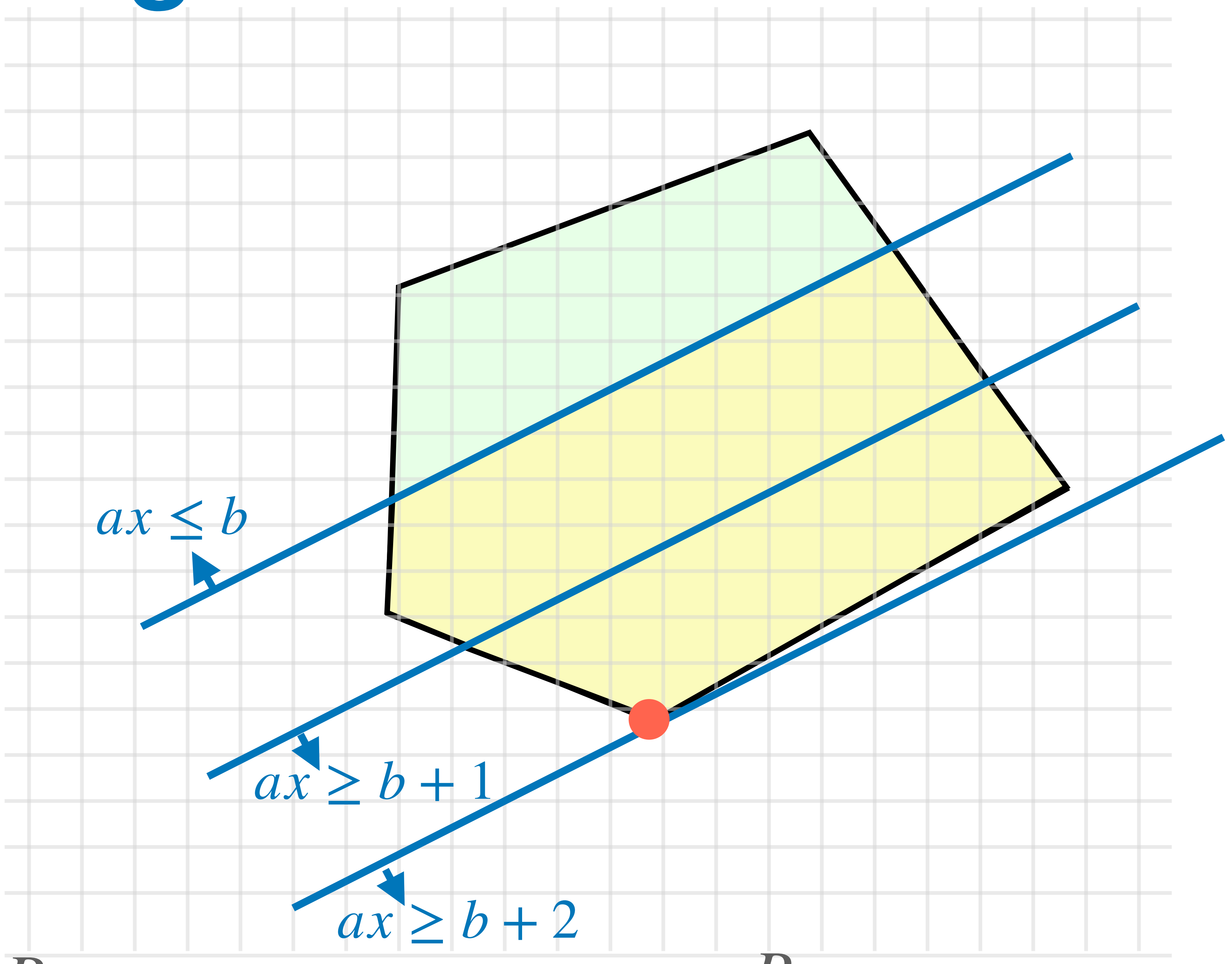

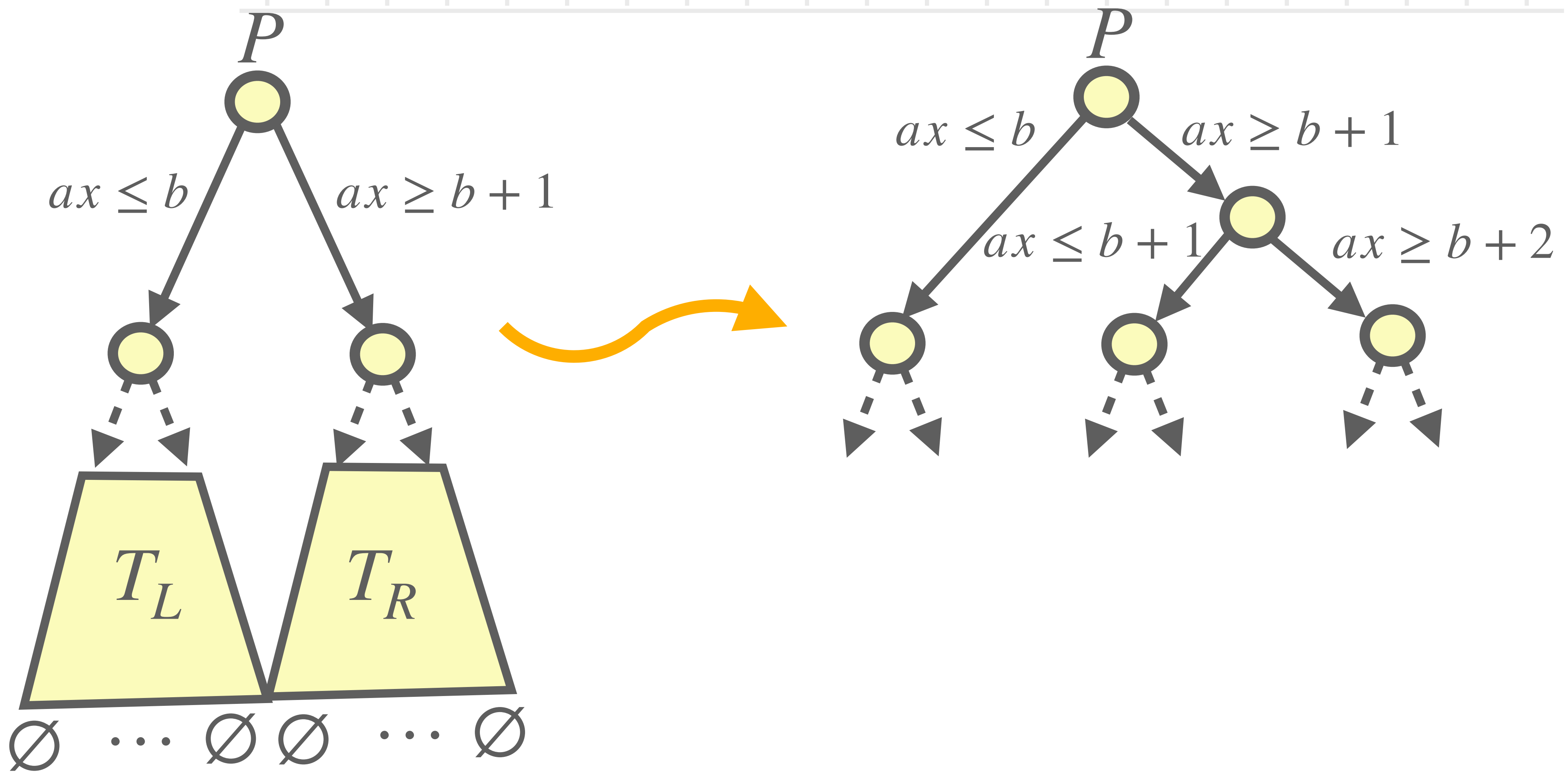

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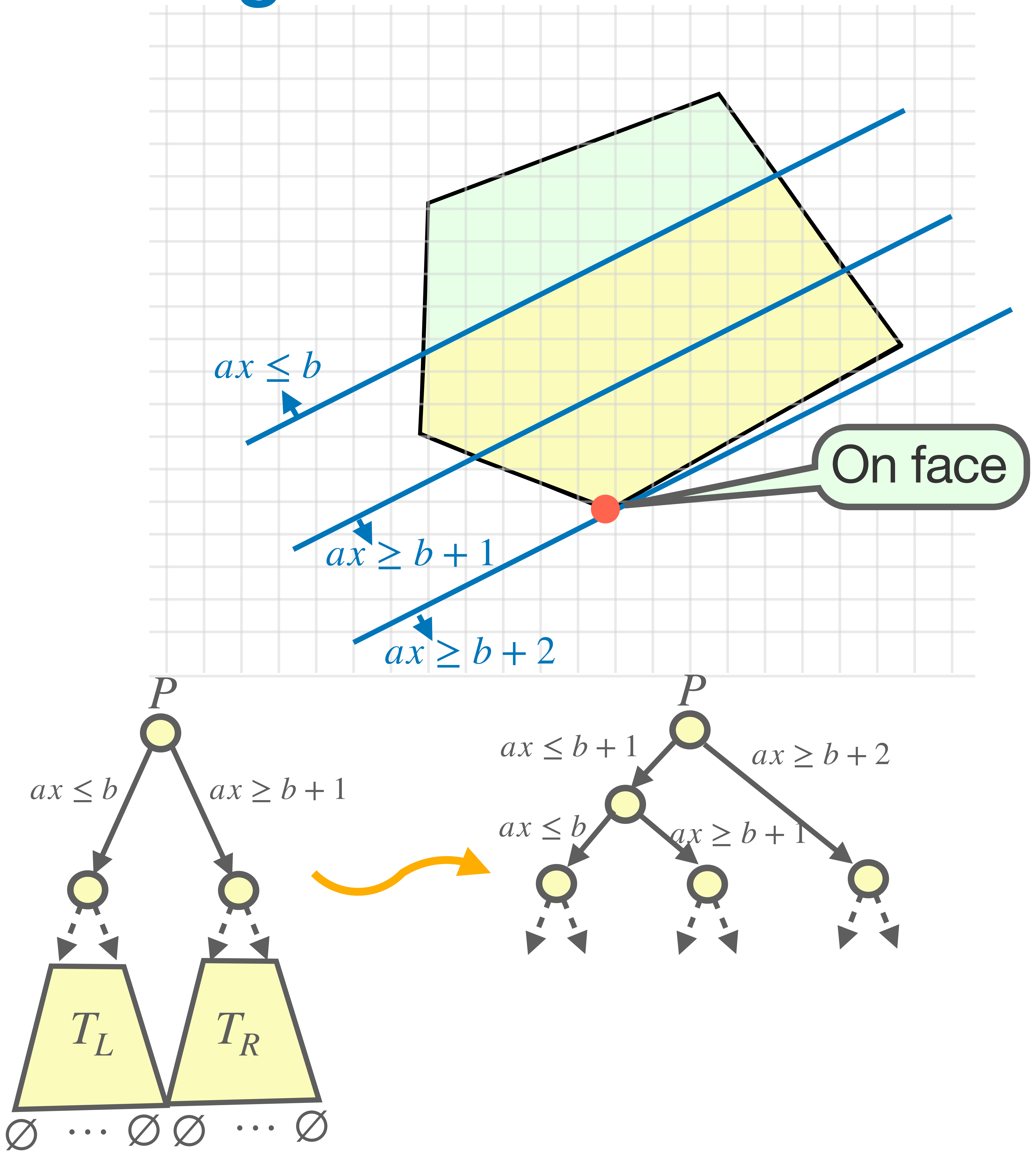

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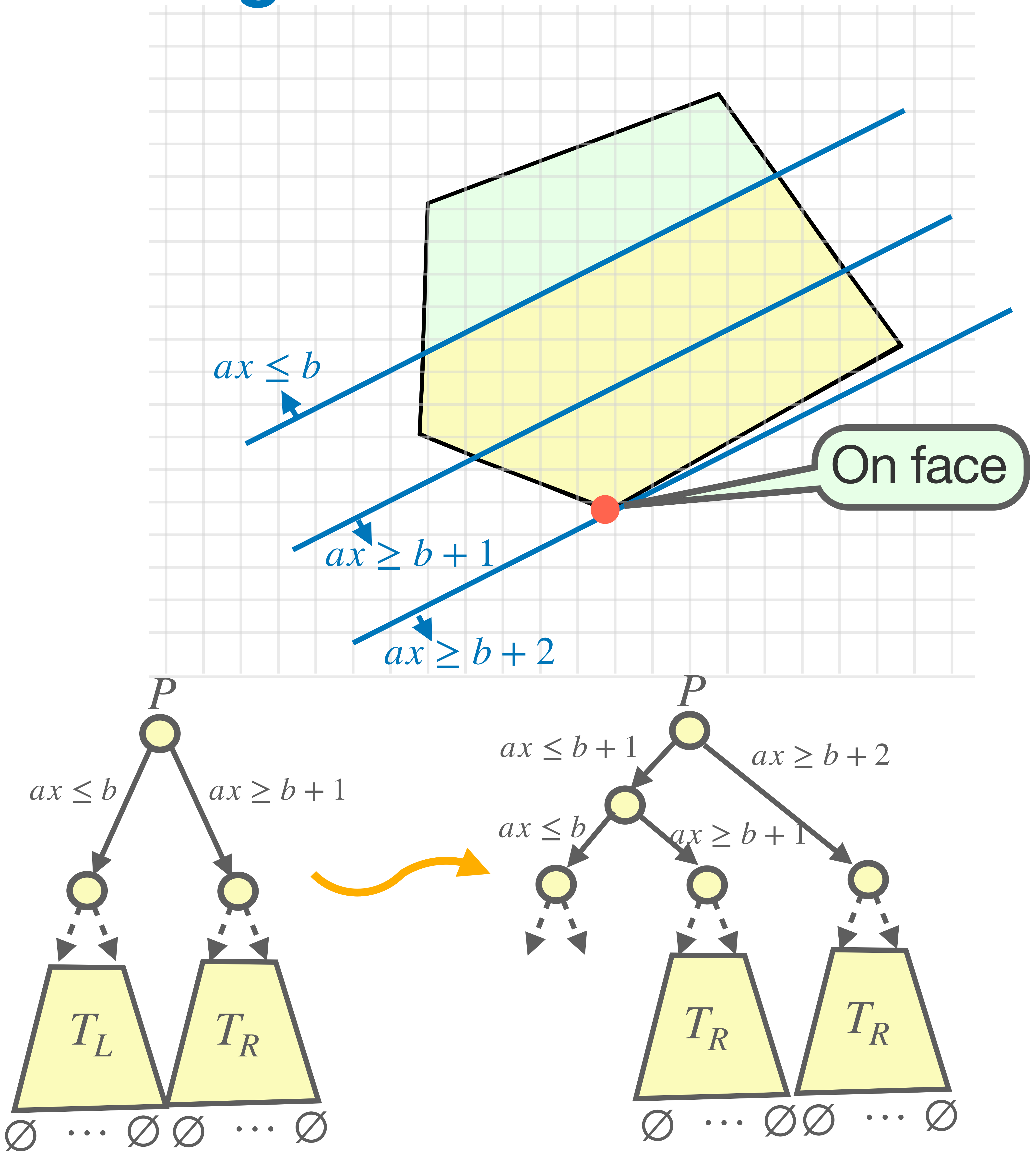

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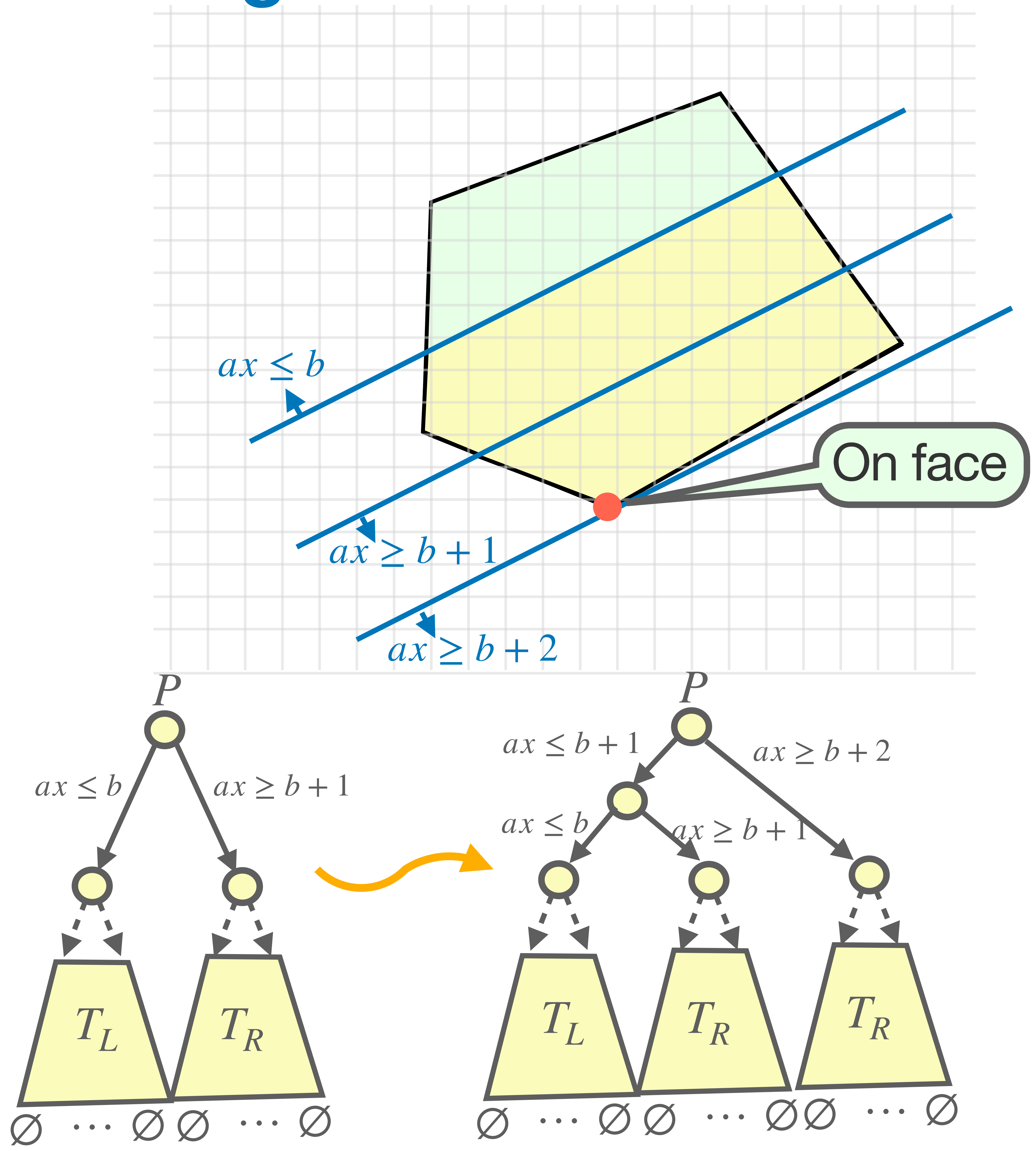

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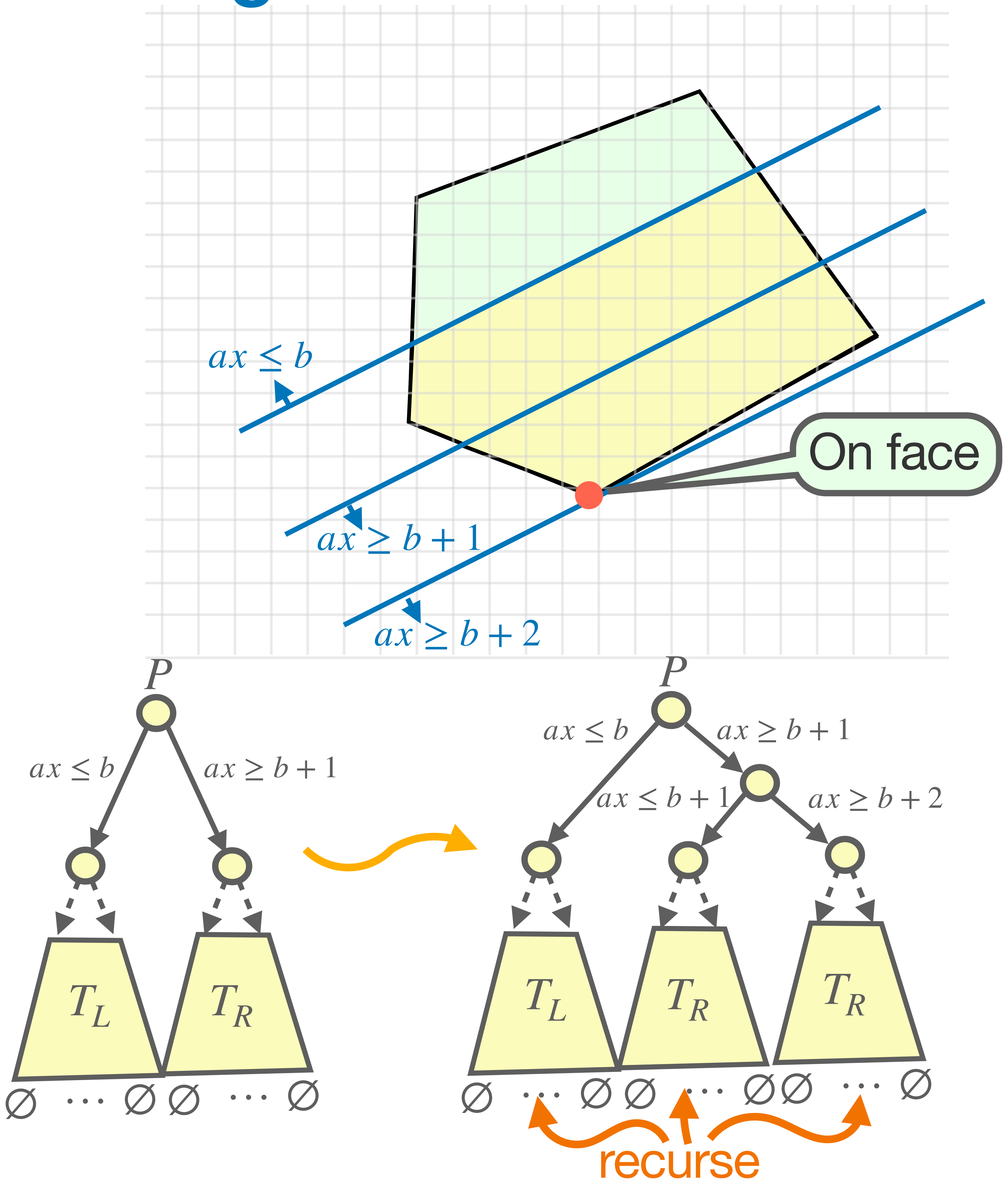

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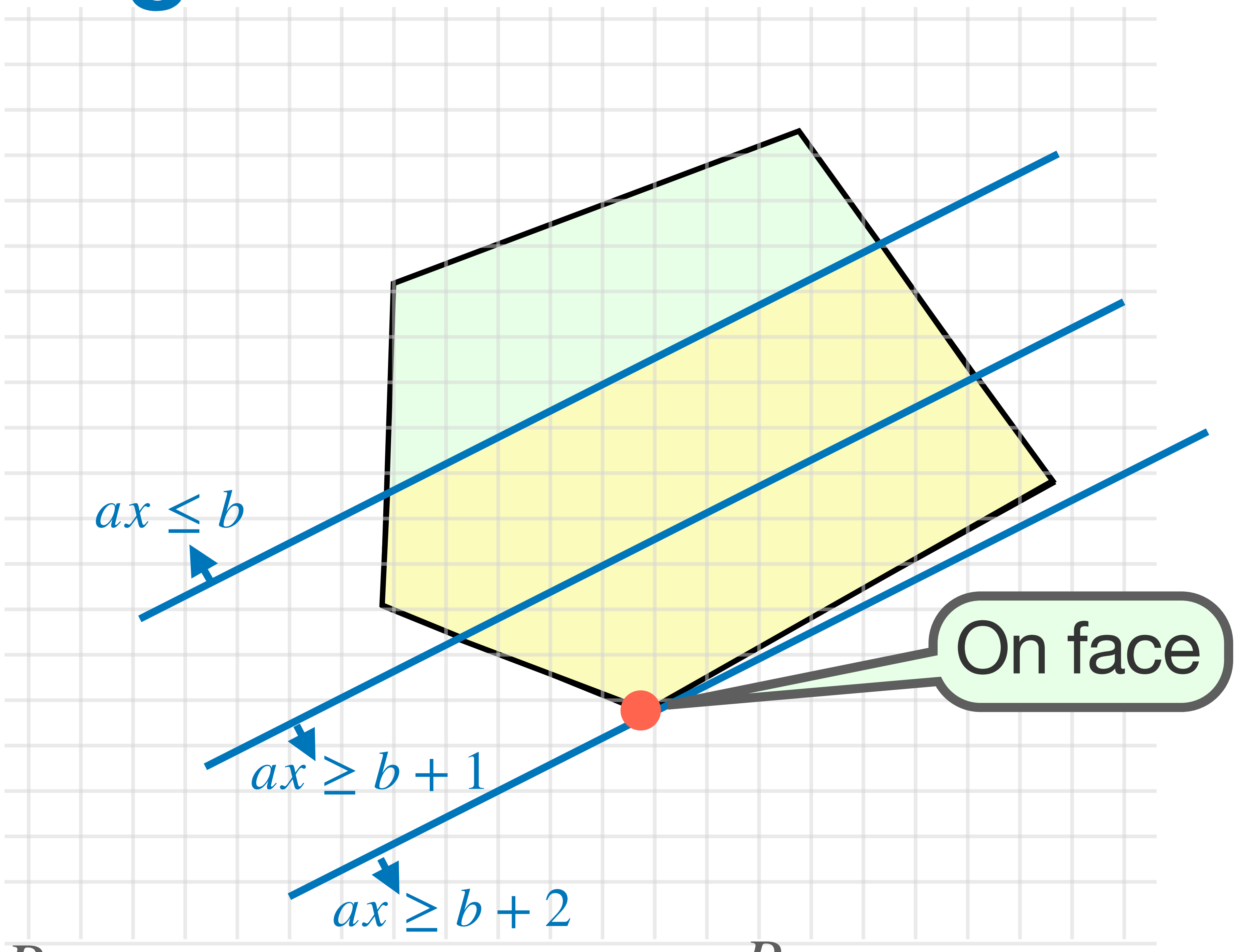
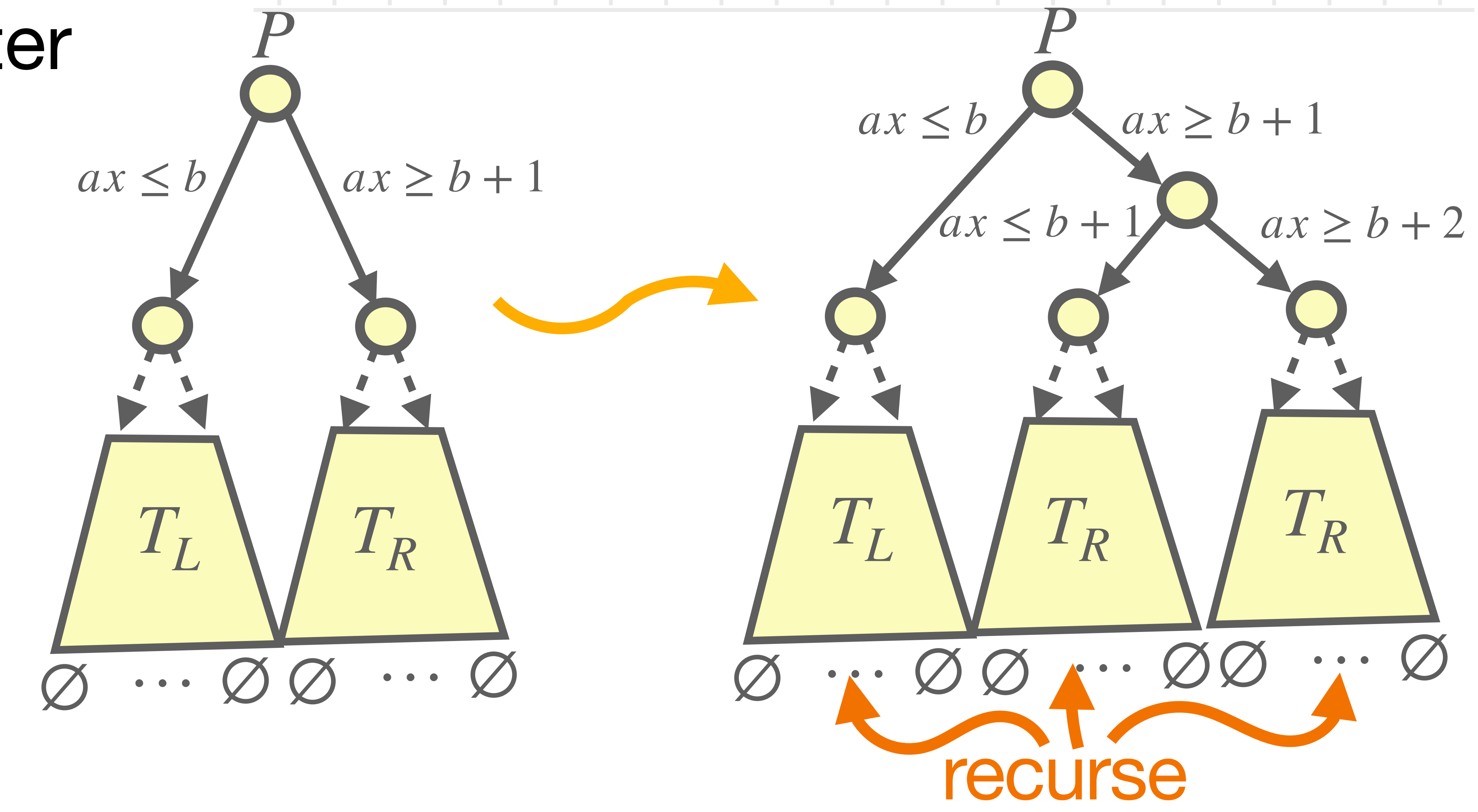
Recursive blowup is proportional to width of slab, diameter of polytope.





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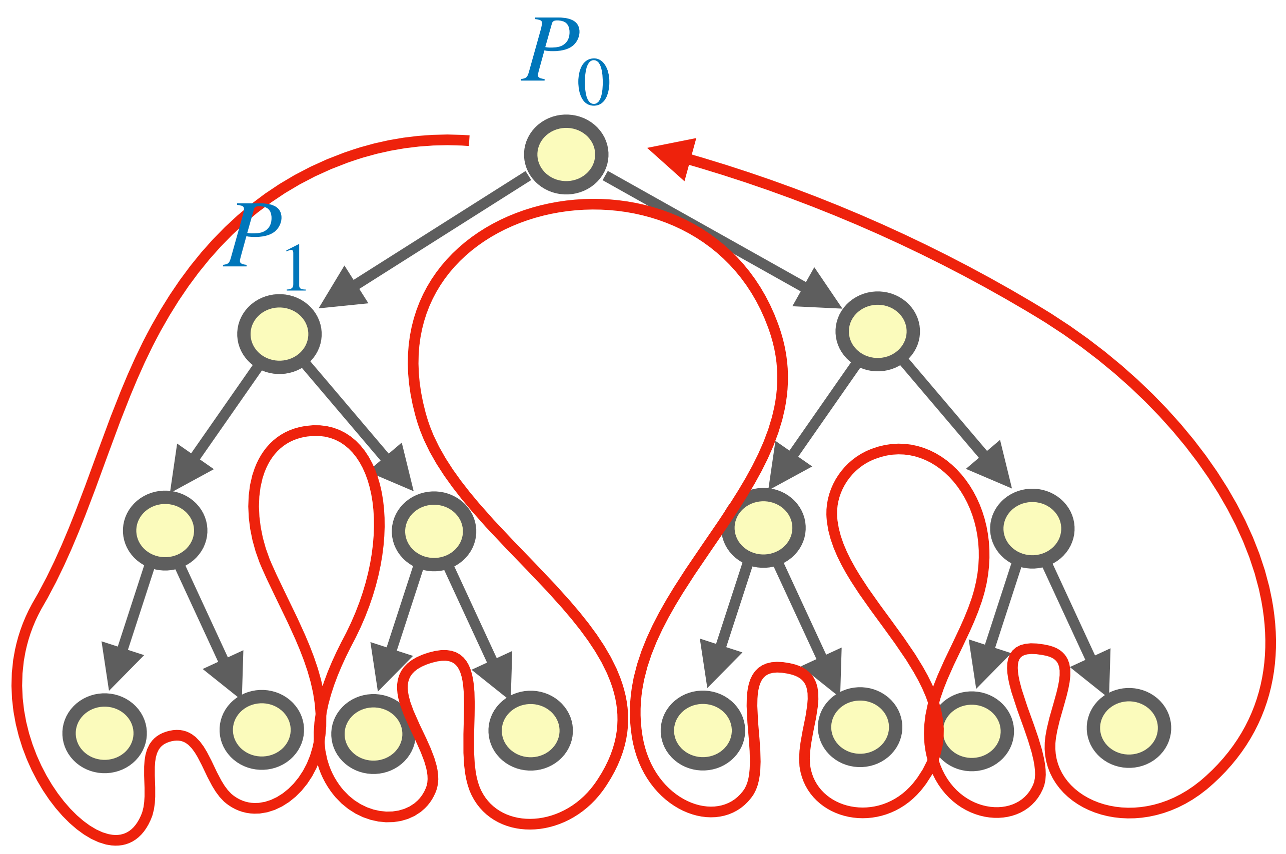

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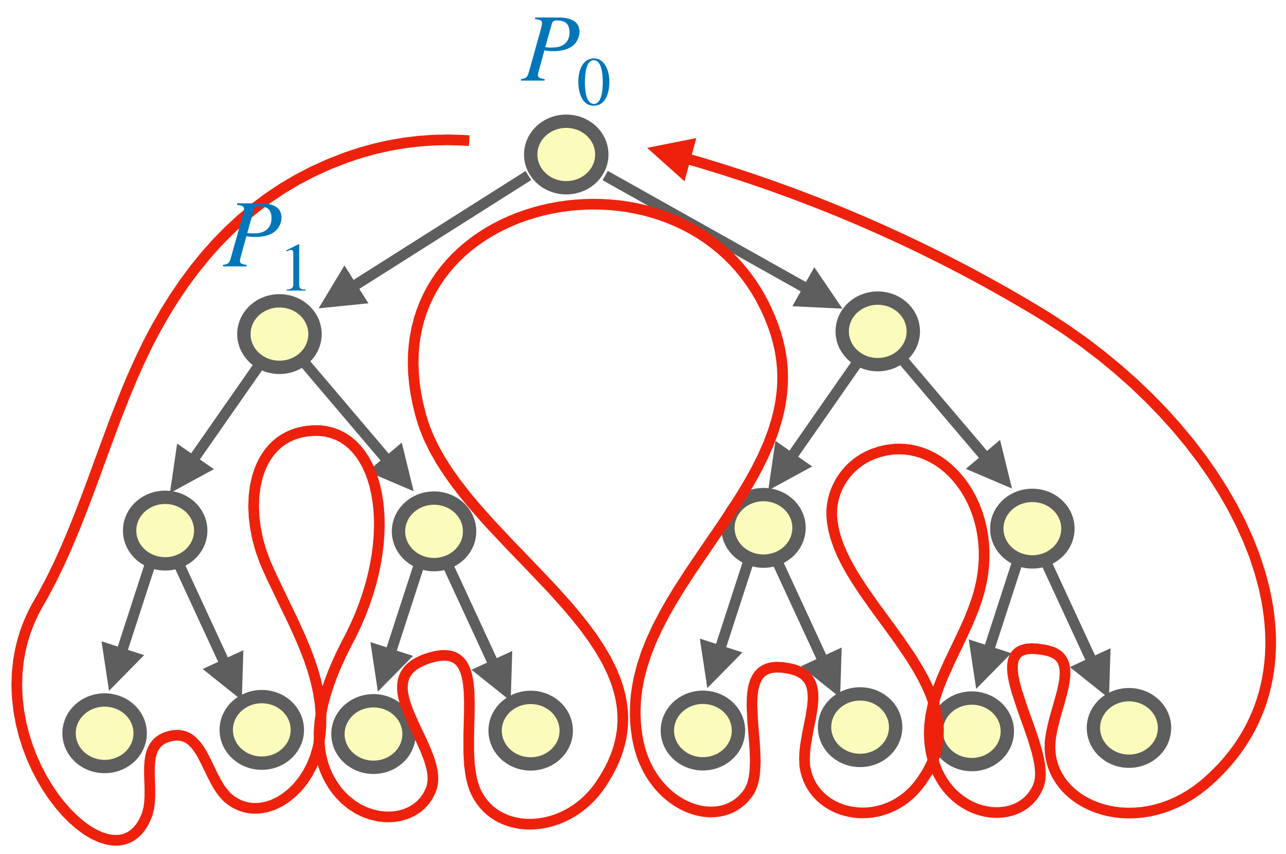

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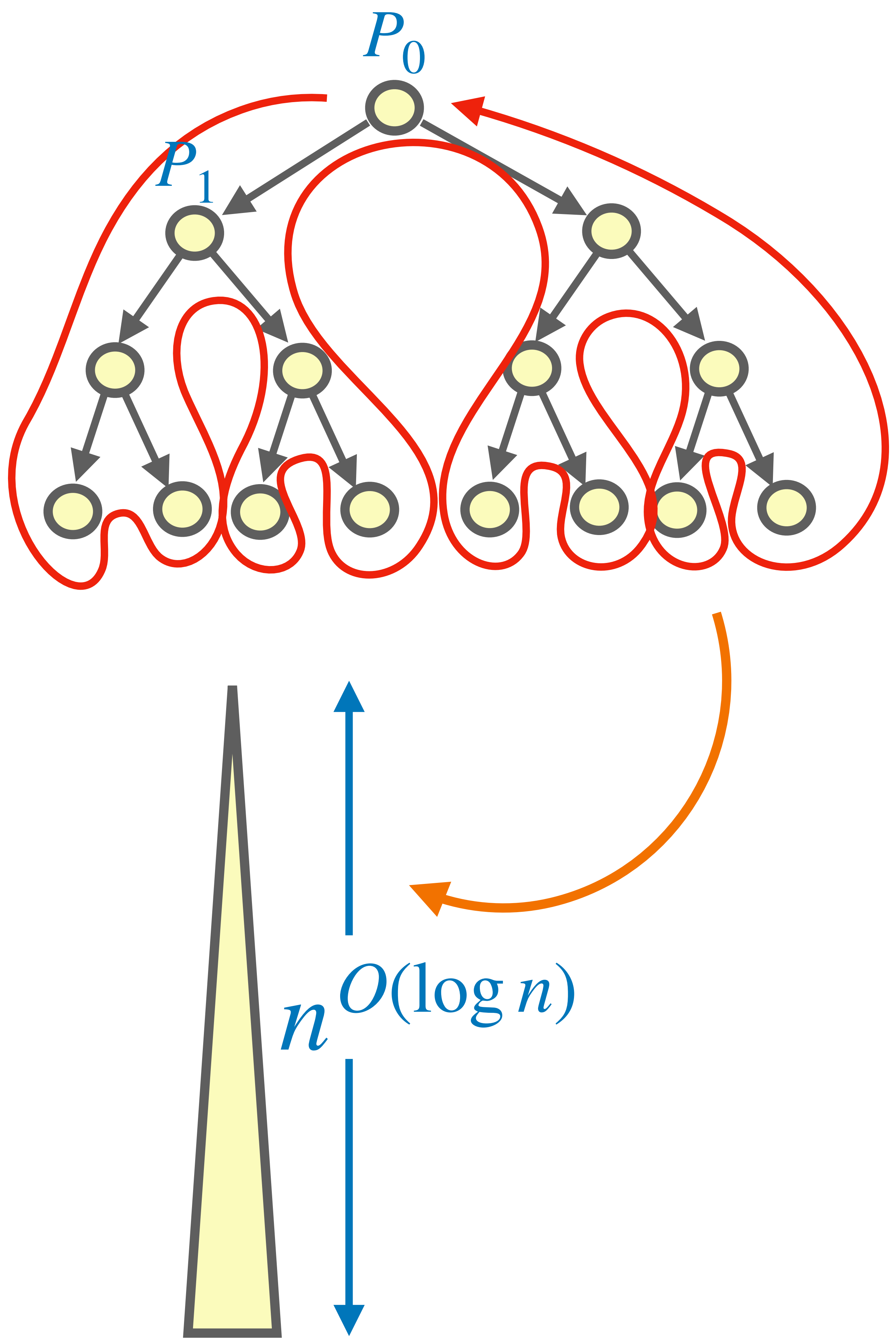

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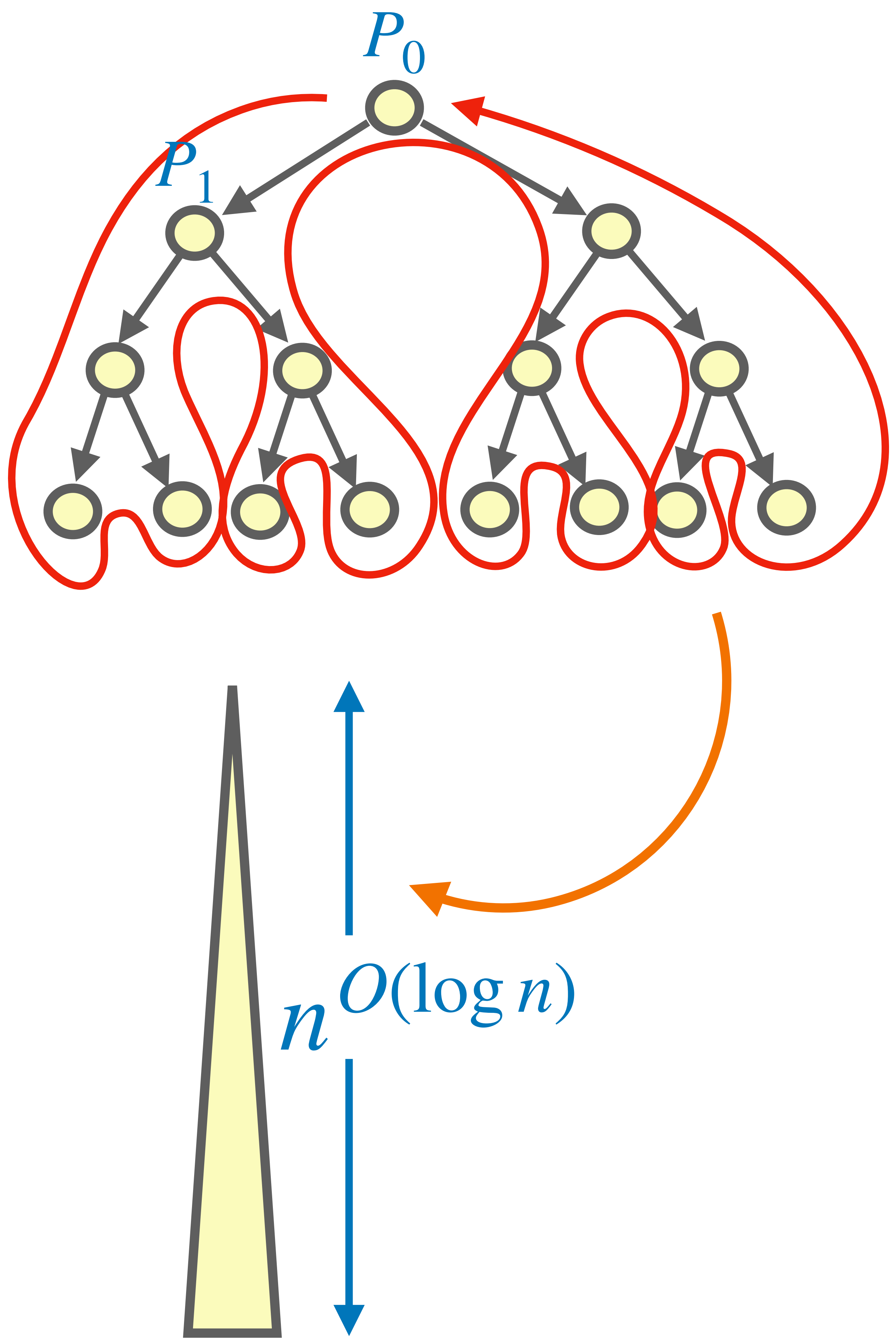

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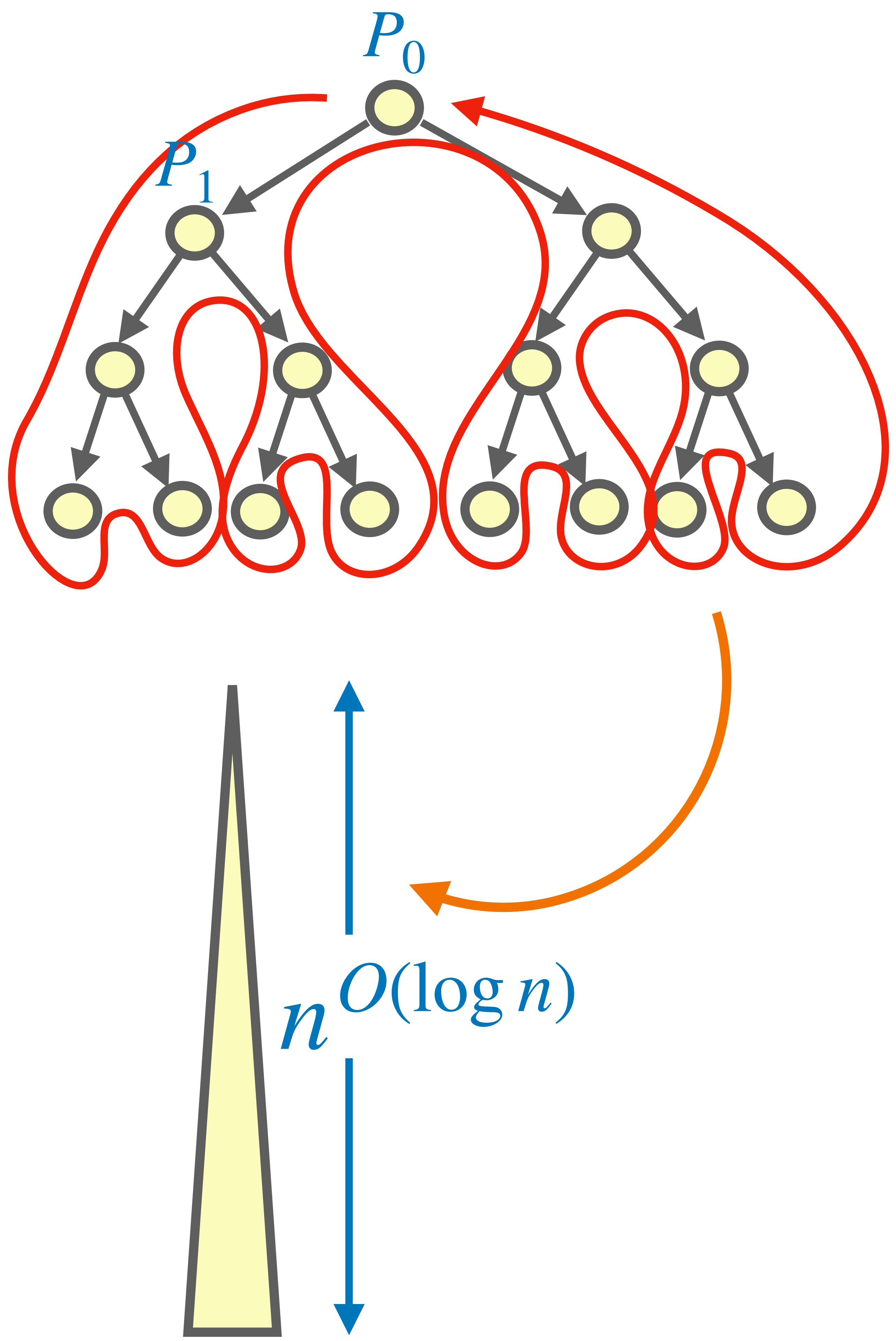

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Builds on [Razborov16]

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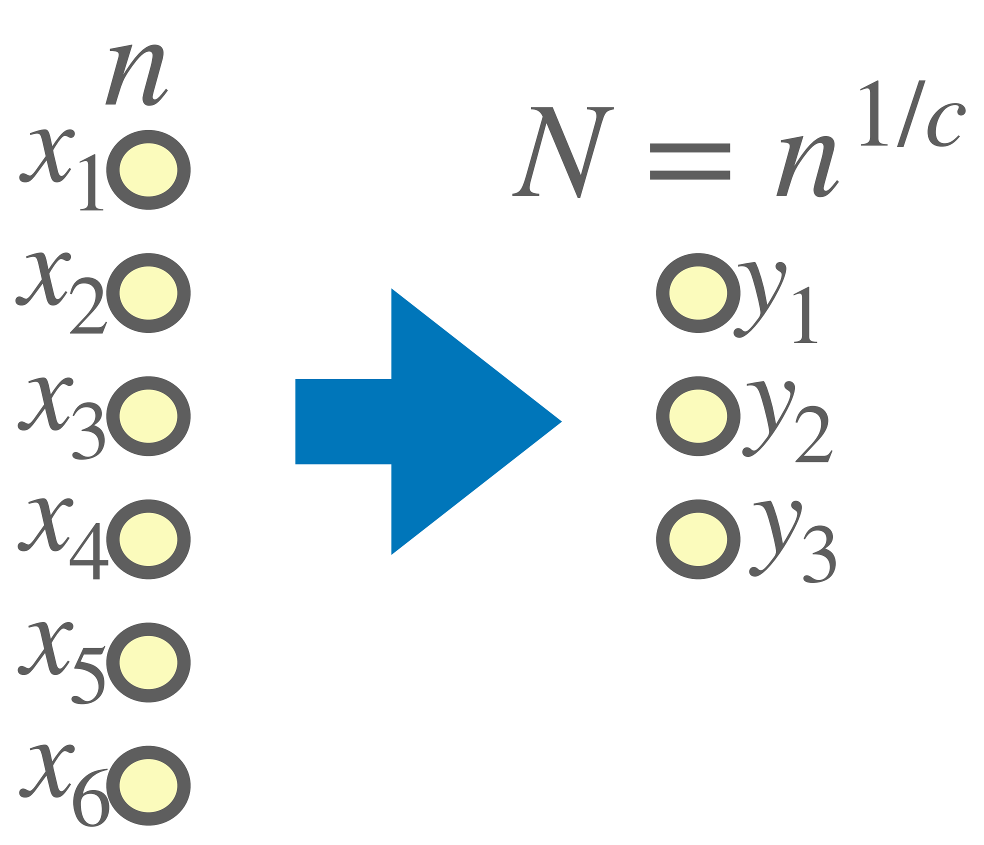

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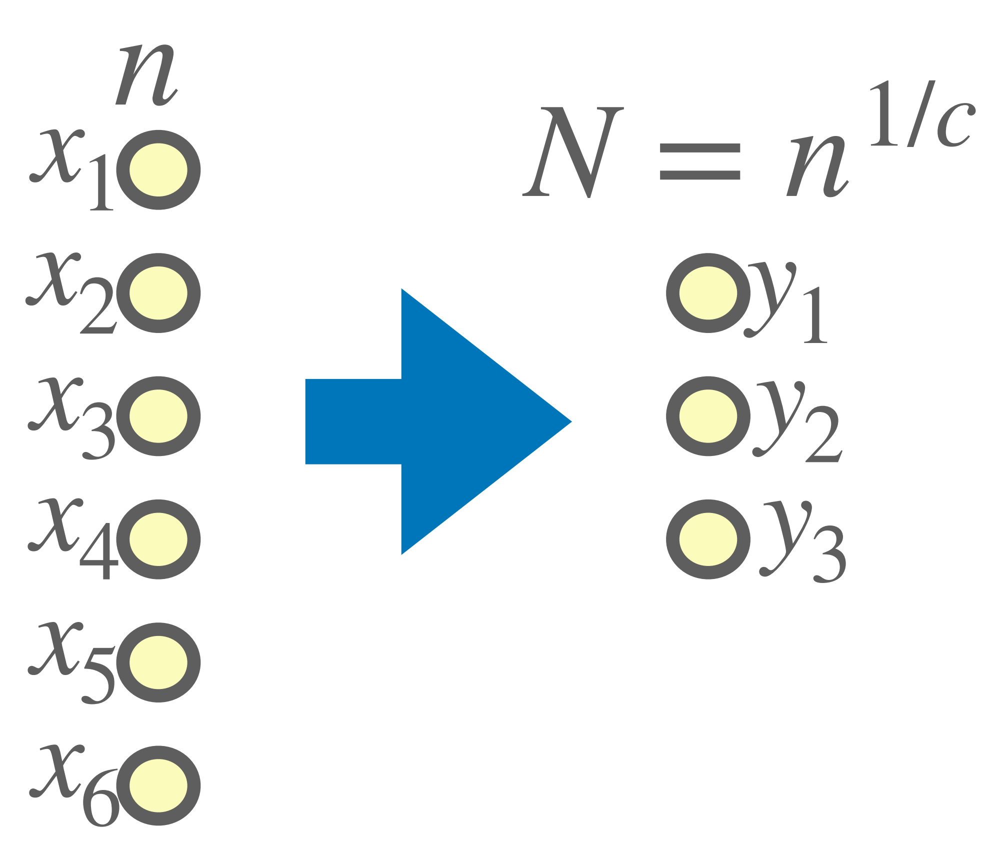

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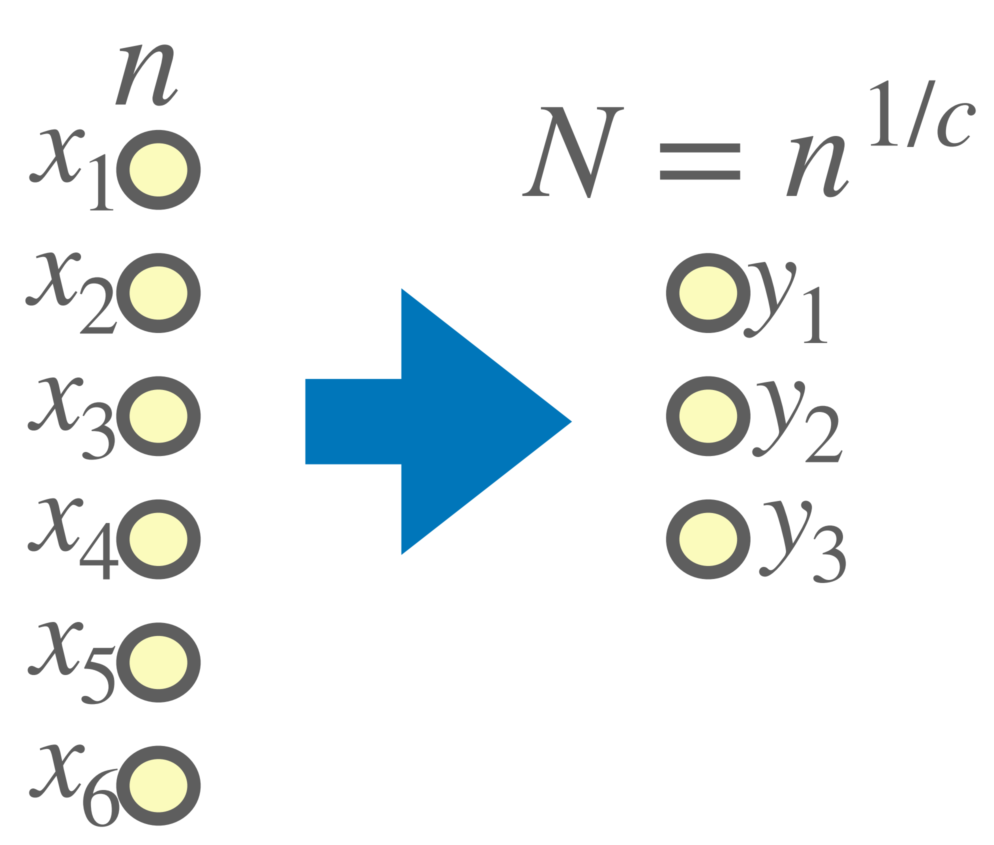

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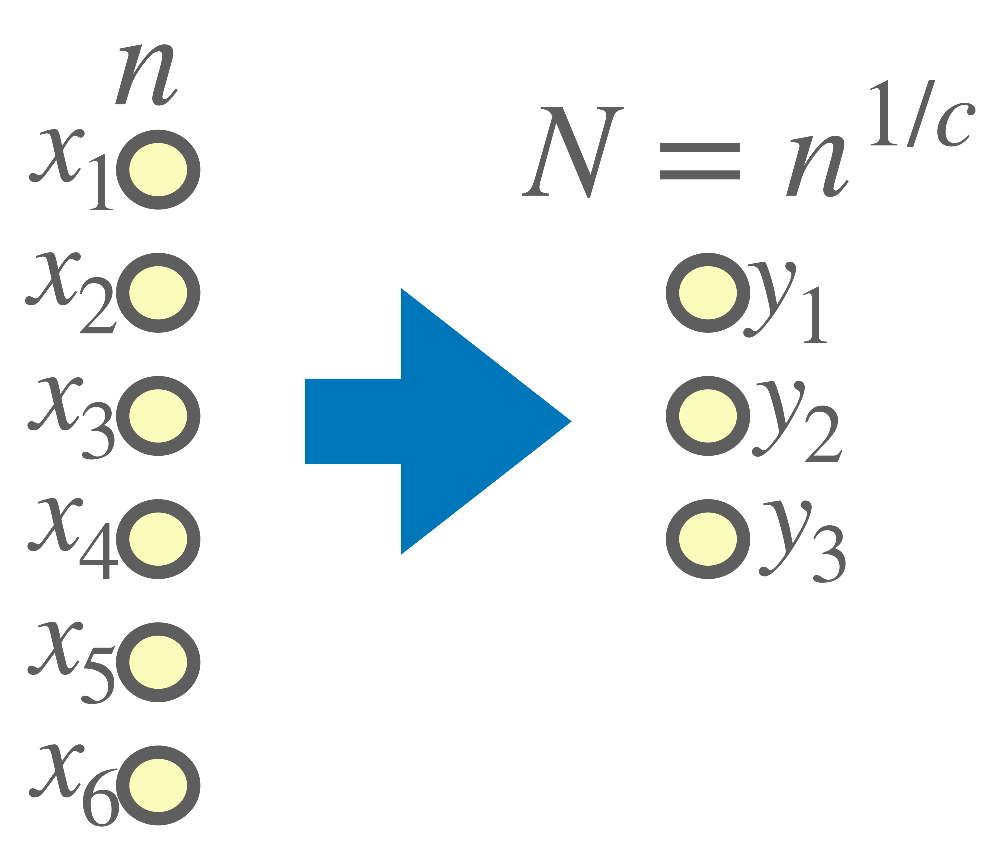

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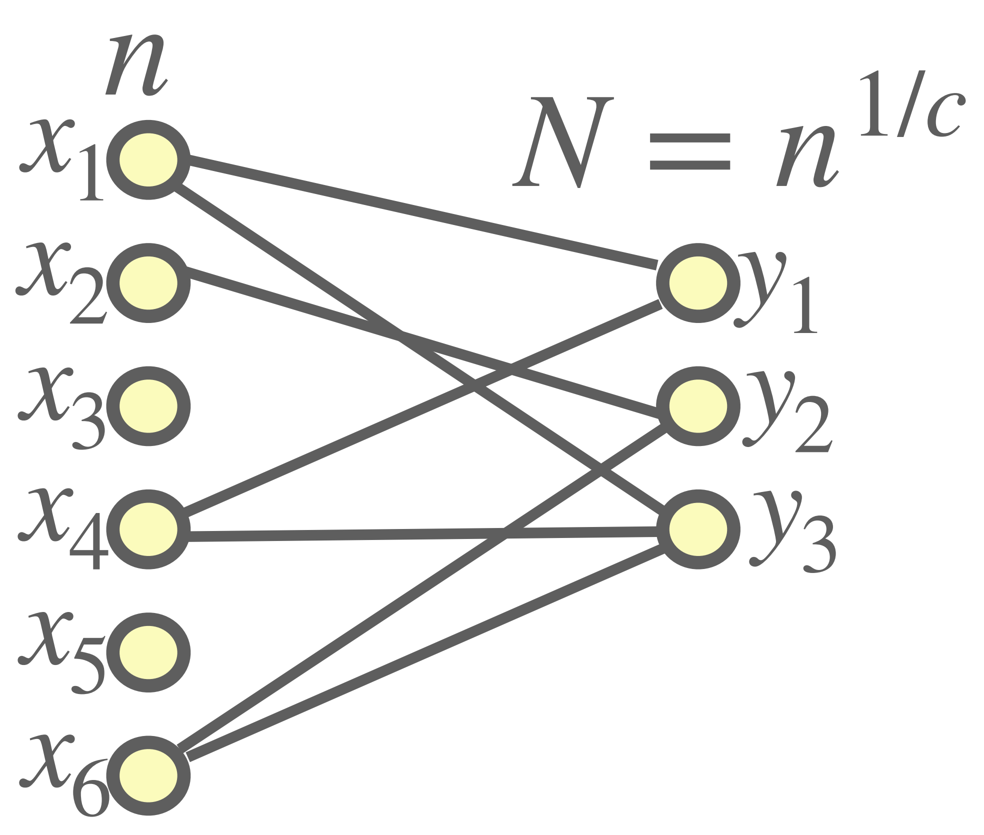

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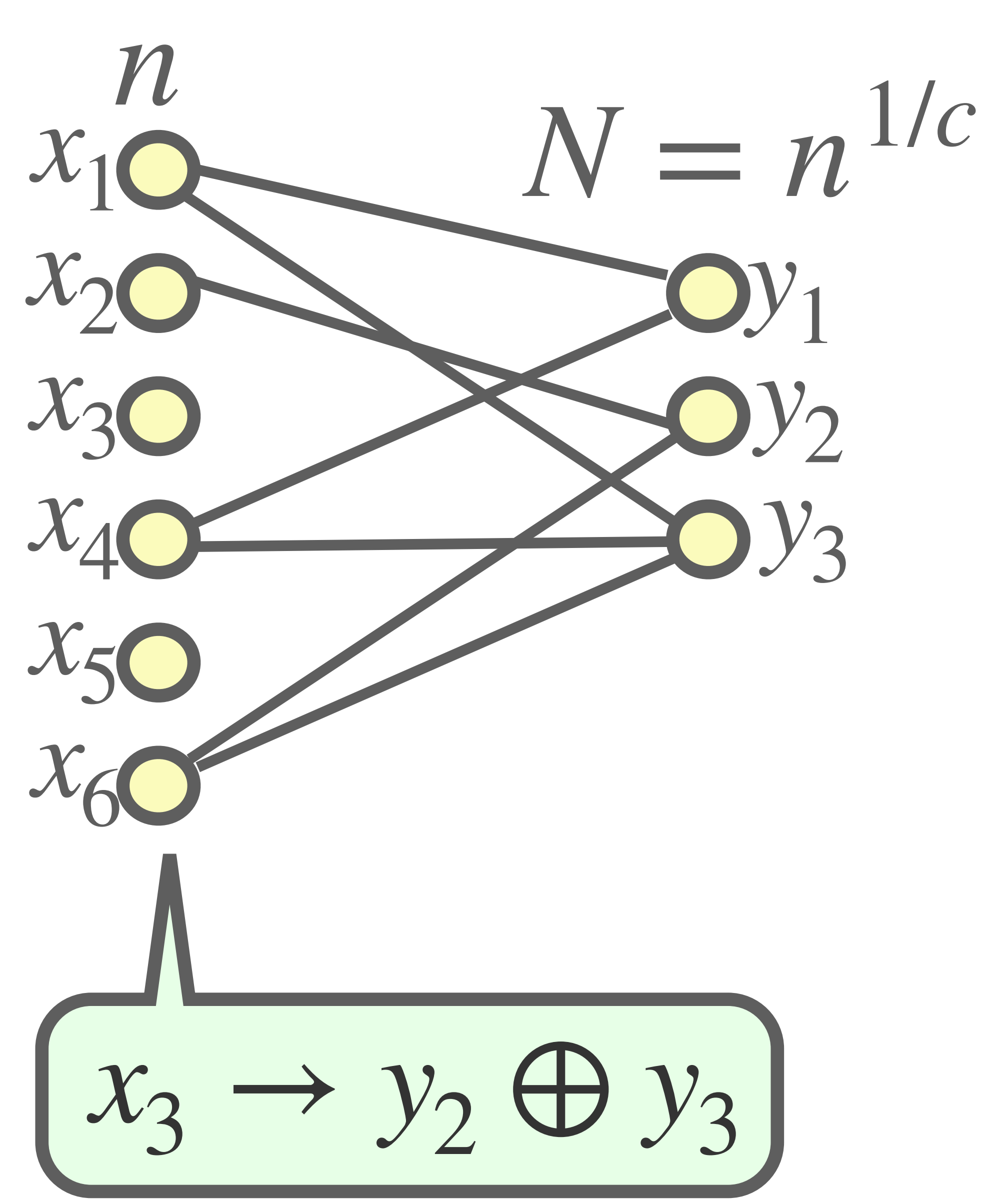

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Tradeoffs for Res(k), Cutting Planes follow by **lifting** the Resolution tradeoff.

Can we use this to obtain the tradeoff for Tseitin?

- Not obviously — Tseitin is exponentially hard for Resolution!

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Theorem [FPR22]: Supercritical size/depth tradeoff for Resolution, Res(k), Cutting Planes.

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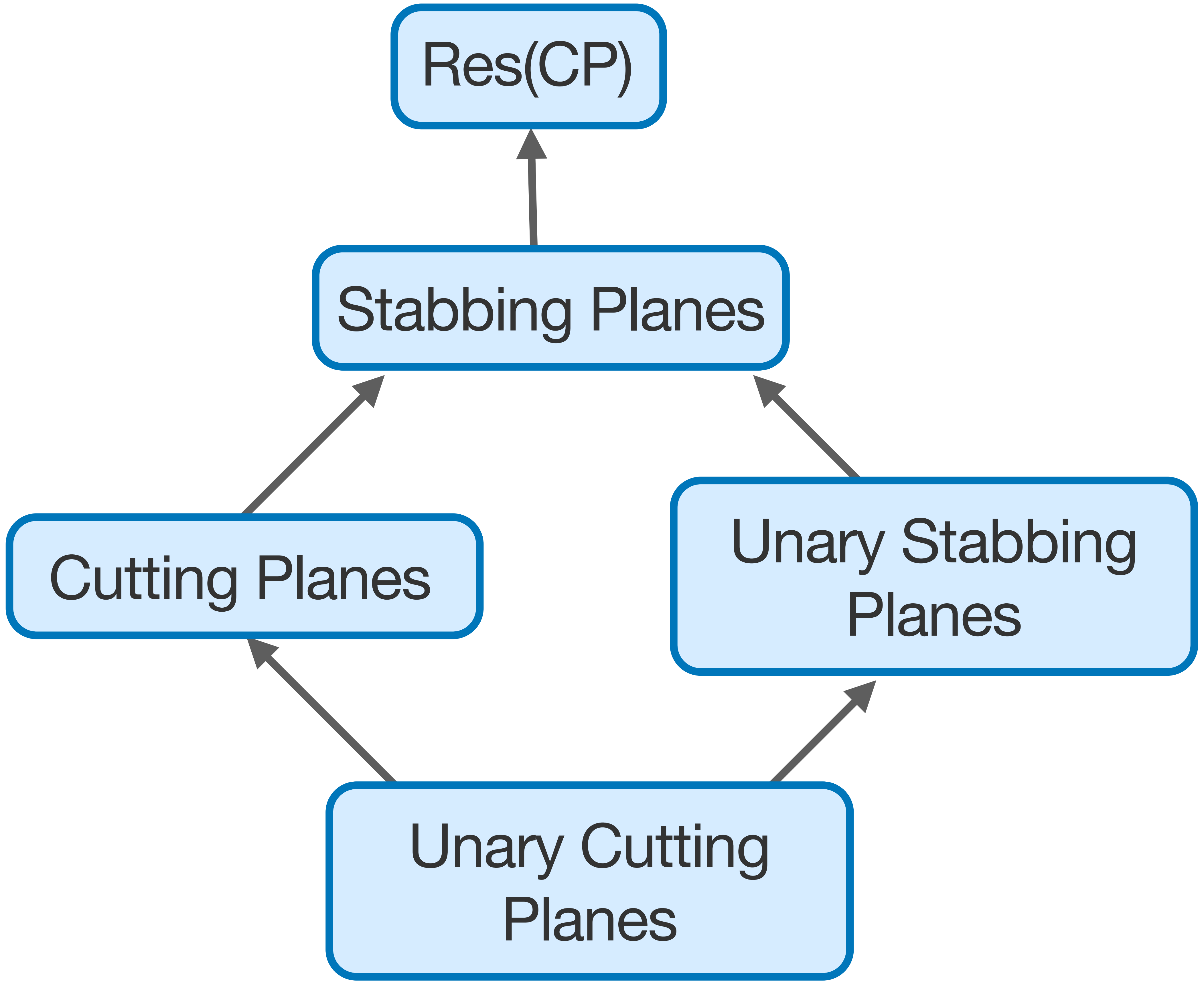
- Uses a geometric argument.

Open Problems

- Prove or disprove the conjecture!
- Can we improve the simulation to **high coefficient** Stabbing Planes?
  - Or, alternatively, can we separate **high coefficient** Stabbing Planes from **low coefficient** Stabbing Planes?
- A generalization of Stabbing Planes to dag-like proofs is called Res(CP).
  - Can Stabbing Planes simulate Res(CP)?
  - [ABE02] Cutting Planes cannot simulate Res(CP)



Thanks!

Shrijver Lemma

Lemma [Schrijver80]: If there is a refutation of a face $P \cap \{ax \leq b\}$ in Cutting Planes then there is a Cutting Planes derivation of $P \cap \{ax \geq b + 1\}$ from P of the same size!

Idea: Since all points in $P \cap \{ax = b\}$ lie on the line $ax \geq b$, we can **rotate** each CG-cut so that it only depends on P and $ax \geq b$ (no longer depends on $ax \leq b$).