SAT Beyond Boolean Interpretations

A. Pavan[1] ⓡ Kuldeep S. Meel[2] ⓡ N. V. Vinodchandran[3*] ⓡ
Arnab Bhattacharyya[2]

[1] Iowa State University
[2] National University of Singapore
[3] University of Nebraska-Lincoln

Simons 2023: Satisfiability: Theory, Practice, and Beyond

* Vinodchandran Variyam

Boolean Interpretations

$F := (x_1) \wedge (x_2) \wedge (\neg x_1 \vee \neg x_2)$

SAT: Given a Boolean formula F. Decide whether there is a truth assignment to the variables so that F is evaluated to True.

Boolean Interpretations

$F := (x_1) \wedge (x_2) \wedge (\neg x_1 \vee \neg x_2)$

SAT: Given a Boolean formula F. Decide whether there is a truth assignment to the variables so that F is evaluated to True.

- Boolean Interpretation
  - $K = \{0, 1\}$
  - $\neg :=$ NOT function
  - $\wedge :=$ AND function
  - $\vee :=$ OR function
  - $0 < 1$ (total order)

SAT: Given a Boolean formula F. Compute $\max_\pi\{\pi(F)\}$ over all interpretations $\pi : X \to K$.

X is the set of variables and $\pi(F)$ is the natural extension of π to F.

F is satisfiable if and only if $\max_\pi\{\pi(F)\} = 1$.

Beyond Boolean Interpretations

$F := (x_1) \wedge (x_2) \wedge (\neg x_1 \vee \neg x_2)$

Viterbi semiring interpretation

- $K = [0, 1]$ (with natural total order)
- $\wedge :=$ MULT function
- $\vee :=$ MAX function
- $\neg x := 1 - x$

Given F: Compute $\max_{\pi}\{\pi(F)\}$ over all interpretations $\pi : X \to K$.

Beyond Boolean Interpretations

$F := (x_1) \wedge (x_2) \wedge (\neg x_1 \vee \neg x_2)$

Viterbi semiring interpretation

- $K = [0, 1]$ (with natural total order)
- $\wedge :=$ MULT function
- $\vee :=$ MAX function
- $\neg x := 1 - x$

Given F: Compute $\max_\pi\{\pi(F)\}$ over all interpretations $\pi : X \to K$.

For F above:

- $\max\{x_1 x_2(1 - x_1), x_1 x_2(1 - x_2)\}$
- $\pi(x_1) = 0.5$; $\pi(x_2) = 1$
- $\pi(F) = 0.5 \cdot 1 \cdot 0.5 = \boxed{0.25}$

Beyond Boolean Interpretations

$F := (x_1) \wedge (x_2) \wedge (\neg x_1 \vee \neg x_2)$

Viterbi semiring interpretation

- $K = [0, 1]$ (with natural total order)
- $\wedge :=$ MULT function
- $\vee :=$ MAX function
- $\neg x := 1 - x$

Given F: Compute $\max_\pi\{\pi(F)\}$ over all interpretations $\pi : X \to K$.

For F above:

- $\max\{x_1 x_2 (1 - x_1), x_1 x_2 (1 - x_2)\}$
- $\pi(x_1) = 0.5$; $\pi(x_2) = 1$
- $\pi(F) = 0.5 \cdot 1 \cdot 0.5 = \boxed{0.25}$

F is satisfiable (Boolean) $\Leftrightarrow \max_\pi\{\pi(F)\} = 1$ (Viterbi)

Interpretation of Negation

How do we interpret $\neg : K \to K$?

$\neg(x) = 1 - x$ is one of them.

For our upper bounds any "reasonable" interpretation of negation suffice.

$$\begin{aligned} \neg\neg(x) &= x \\ \neg(0) &= 1 \end{aligned}$$

Useful Semirings

- Viterbi semiring $\mathbb{V} = ([0,1], \max, \cdot, 0, 1)$.
  - Database provenance, where $x \in [0,1]$ is interpreted as a *confidence score*.
  - Probabilistic parsing, probabilistic CSPs, Hidden Markov Models.
- Fuzzy semiring $\mathbb{F} = ([0,1], \max, \min, 0, 1)$.
- Access control semiring $\mathbb{A}_k = ([k], \max, \min, 0, k)$
  - Security Specification. Each $i \in [k]$ is associated with a access control level with natural ordering. 0 corresponds to public access and k corresponds to no access at all.

Computational Problem OptSemVal

Fix a semiring K of your choice.

OptSemVal:
Given a formula F (in negation normal form), compute $\max_\pi\{\pi(F)\}$ over all interpretations $\pi : X \to K$.

Computational Problem OptSemVal

Fix a semiring K of your choice.

OptSemVal:
Given a formula F (in negation normal form), compute $\max_\pi\{\pi(F)\}$ over all interpretations $\pi : X \to K$.

What is the complexity of OptSemVal?

Viterbi Interpretations

$$\mathrm{FP}^{\mathrm{NP[log]}} \quad \leq \quad \mathrm{OptSemVal} \quad \leq \quad \mathrm{FP}^{\mathrm{NP}}.$$

OptSemVal over other Semirings

Fuzzy, Access Control $\equiv$ Boolean (SAT)

Hardness for Viterbi

MaxSAT $\leq$ OptSemVal

Hardness for Viterbi

MaxSAT $\leq$ OptSemVal

Confidence Bounding Lemma: Let F be a CNF formula with m clauses and r the maximum number of satisfiable clauses (over the Boolean semiring). Then for any (Viterbi) interpretation π

$$\pi(F) \leq \frac{1}{4^{m-r}}$$

Hardness for Viterbi

MaxSAT $\leq$ OptSemVal

Confidence Bounding Lemma: Let F be a CNF formula with m clauses and r the maximum number of satisfiable clauses (over the Boolean semiring). Then for any (Viterbi) interpretation π

$$\pi(F) \leq \frac{1}{4^{m-r}}$$

Reduction $F \to F'$: $C_i \to (C_i \vee y_i)$ for each i and add m additional unit clauses $\neg y_i$.

Hardness for Viterbi

MaxSAT $\leq$ OptSemVal

Confidence Bounding Lemma: Let F be a CNF formula with m clauses and r the maximum number of satisfiable clauses (over the Boolean semiring). Then for any (Viterbi) interpretation π

$$\pi(F) \leq \frac{1}{4^{m-r}}$$

Reduction $F \to F'$: $C_i \to (C_i \vee y_i)$ for each i and add m additional unit clauses $\neg y_i$.

Claim: $\text{OptSemVal}(F') = 1/4^{m-r}$ where r is the maximum number of satisfiable clauses for F.

- We can give an interpretation π so that $\pi(F') = 1/4^{m-r}$
- That is the best possible since
  - number of clauses of $F' = 2m$
  - maximum number of clauses that can be satisfied is $m + r$

Upperbound - Overview

OptSemVal $\in$ FP^{NP}

Define a binary search language $L_{opt} = \{\langle F, v\rangle \mid \text{OptSemVal}(F) \geq v\}$.

- Perform binary search over $[0, 1]$ by making queries to L_{opt}

Upperbound - Overview

$\mathsf{OptSemVal} \in \mathsf{FP}^{\mathsf{NP}}$

Define a binary search language $L_{opt} = \{\langle F, v\rangle \mid \mathsf{OptSemVal}(F) \geq v\}$.

- Perform binary search over $[0, 1]$ by making queries to L_{opt}

  Challenge: OptSemVal(F) could potentially be any real number. Do not know when to stop the binary search.

Upperbound - Overview

OptSemVal $\in \mathsf{FP}^{\mathsf{NP}}$

Define a binary search language $L_{opt} = \{\langle F, v\rangle \mid \text{OptSemVal}(F) \geq v\}$.

- Perform binary search over $[0,1]$ by making queries to L_{opt}

  Challenge: OptSemVal(F) could potentially be any real number. Do not know when to stop the binary search.

  Claim: OptSemVal$(F) \in \mathcal{F}_N$ for $N \in 2^{\mathrm{poly}(size(F))}$.

  $\mathcal{F}_N$: Farey Sequence of order N. Fractions of the form A/B, where $1 \leq A, B \leq N$ and $\gcd(A, B) = 1$.

Upperbound - Overview

OptSemVal $\in \mathrm{FP}^{\mathrm{NP}}$

Define a binary search language $L_{opt} = \{\langle F, v\rangle \mid \text{OptSemVal}(F) \geq v\}$.

- Perform binary search over $[0, 1]$ by making queries to L_{opt}

  Challenge: OptSemVal(F) could potentially be any real number. Do not know when to stop the binary search.

  Claim: OptSemVal(F) $\in \mathcal{F}_N$ for $N \in 2^{\mathrm{poly}(size(F))}$.

  $\mathcal{F}_N$: Farey Sequence of order N. Fractions of the form A/B, where $1 \leq A, B \leq N$ and $\gcd(A, B) = 1$.

Other technical challenges exist (eg: $L_{opt} \in \mathrm{NP}$). But we can brute-force through them for the idea to work.

Approximation of OptSemVal over Viterbi

3CNF formulae with m clauses

- Upperbound: There is a polynomial-time, 0.716^m-approximation algorithm.
  - A random interpretation achieves this.

Approximation of OptSemVal over Viterbi

3CNF formulae with m clauses

- Upperbound: There is a polynomial-time, 0.716^m-approximation algorithm.
  - A random interpretation achieves this.
- Hardness: No polynomial-time 0.845^m-approximation algorithm unless $P = NP$.

Future Work

Technical Questions

- Can we close the gap between lower and upper bound for Viterbi?
- Efficient implementation of the $\mathrm{FP}^{\mathrm{NP}}$ algorithm using SAT solvers?

Future Work

Technical Questions

- Can we close the gap between lower and upper bound for Viterbi?
- Efficient implementation of the $\mathrm{FP}^{\mathrm{NP}}$ algorithm using SAT solvers?

Non-technical Question

Explore usefulness of optimization problems that are introduced in other areas (reasoning about neural networks?).

Increasing interest in Beyond-Boolean interpretations in databases.

Simons Program Fall 2023: Logic and Algorithms in Database Theory and AI

One of the themes: "Extensions of logics to semirings and aggregation."

Thank you!

Work appeared in AAAI 2023