MCSAT BASED APPROACHES FOR NON-LINEAR MODULAR ARITHMETIC

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Satisfiability: Theory, Practice, and Beyond

Extended Reunion: Satisfiability

Simons Institute, Berkeley, CA, US

April 20, 2023

Modulo 5

$$x^2 - 1 = 0$$

$$xy - y - 1 = 0$$

$$xy - 2 \neq 0$$

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$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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$$x \mapsto 4, y \mapsto 2$$

Solving polynomial equations

Solving polynomial equations is one of the oldest and hardest problem in mathematics.

Algebraic closed fields: decidable (Gröbner bases)

Finite domains are not algebraically closed

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Non-linear polynomial reasoning over finite domains is currently of interest in automated reasoning over cryptosystems:

- Finite field $\mathbb{F}_q[X]$: Zero-knowledge proofs, elliptic curve cryptography
- $\mathbb{Z}/2^k\mathbb{Z}[X]$: Bit-vector solving, e.g. in smart contracts

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Computer Algebra

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- Powerful in finding all solutions
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We are typically not interested in finding all solutions!

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Model Constructing Satisfiability (MCSat) **[Jovanović et al., VMCAI'13]**

- Finding satisfiable instances of polynomial arithmetic.
- Combines CDCL-style search with algebraic decompositions.

MCSAT based approaches for non-linear modular arithmetic

1. Constraints in $\mathbb{F}_q[X]$

- Finite field
- Not algebraically closed
- Constraints: $=, \neq$

Modulo 5

$$\begin{aligned} x^2 - 1 &= 0 \\ xy - y - 1 &= 0 \\ xy - 2 &\neq 0 \end{aligned}$$

$\Rightarrow$ FFSat
Thomas Hader, Daniela Kaufmann,
Laura Kovács

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2. Constraints in $\mathbb{Z}/2^k\mathbb{Z}[X]$

- Finite commutative ring
- Not algebraically closed
- Constraints: $=, \neq, <, >, \Omega^*(x, y)$

Modulo 2^4

$$xy + y \leq y + 3$$
$$2y + z = 10$$
$$3x + 6yz + 3z^2 = 1$$

$\Rightarrow$ PolySat

Nikolaj Bjørner, Clemens Eisenhofer, Daniela Kaufmann, Laura Kovács, Jakob Rath

MCSat

MCSat **[Jovanović et al., VMCAI'13]**

- abstract CDCL decision procedure
- integrates theory reasoning in the boolean search engine
- incrementally constructs model while searching
- propagated literals are justified by model assignments

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Successfully applied in the theories of

- non-linear arithmetic constraints over reals **[Jovanović et al., IJCAR'12]**
- linear integer constraints **[Jovanović et al., CADE'11]**
- bitvectors **[Zeljić et al., SAT'16]**

MCSat - Idea

From a given set of clauses $\mathcal{C}$, generate a trail Γ with decided and propagated literals and theory variable assignments that leads to one of the two terminal states $\textsc{unsat}$ or $\textsc{sat}$.

Polynomial system is a set of unit clauses.

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Main components:

- Trail Γ records assignments and reasons
- For each variable x, keep track of viable values V_x
- Conflict C: set of constraints that contradicts Γ
- Conflict analysis learn a new constraint to avoid the conflict in the future

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Variables $x_1 < x_2 < \ldots < x_n$

$$\Gamma = [\![F_1, \ldots, F_l, x_1 \mapsto \alpha_1, G_1, \ldots, G_m, x_2 \mapsto \alpha_2, H_1, \ldots, H_n, \ldots]\!]$$

literals F_i over $[x_1]$, G_i over $[x_1, x_2]$, H_i over $[x_1, x_2, x_3]$.

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Regular **boolean propagation**:

Clause $\mathcal{C}_1 = \{\neg F_1, \neg G_2, H\}$
Add literal H to the trail with justification $\mathcal{C}_1$.

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In addition **theory propagation**:

> **Idea**: From theory (i.e. variable assignments) we know that literal H can't hold, $\neg H$ can be propagated.
> Generate explanation clause E that justifies $\neg H$.

Example: Polynomial Equations

$$C_1: \quad x^2 - 1 = 0 \mod 5$$
$$C_2: \quad xy - y - 1 = 0 \mod 5$$

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3. $\Gamma = [\![(x^2 - 1 = 0)^\delta, (x \mapsto 1)^{C_1,\delta}, (xy - y - 1 = 0)]\!]$ — add C_2

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3. $\Gamma = [\![(x^2 - 1 = 0)^\delta, (x \mapsto 1)^{C_1,\delta}, (xy - y - 1 = 0)]\!]$ — add C_2
   $\rightsquigarrow C_2|_\Gamma : \ -1 = 0$
   Conflict: $\mathcal{C} = \{C_2, x = 1, C_1\}$
   Generate explanation clause $E = \{x + 1 = 0, \neg C_2\}$ using theory propagation.
   To satisfy C_2 we resolve using E and backtrack to assign a different value to x.

Explain Function

Informal: Bring theory knowledge into the search procedure on demand.

Key ingredient for every MCSat procedure is the explain function!

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Example

For $q = 5$ the field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.

- $(2 \cdot 3) + 4 = 0$
- inverse of 2 is 3, as $2 \cdot 3 = 1$

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General Idea: Given a trail Γ

$$\Gamma = [\![\ldots, x_{k-1} \mapsto \alpha_{k-1}, F_1, F_2, \ldots, F_l]\!]$$

for $1 \leq i \leq l$: $x_k \in \mathsf{var}(F_i)$ and $\exists \alpha_k \in \mathbb{F}_q$ s.t. $\nu[\Gamma][x_k \mapsto \alpha_k](F_i) = \mathsf{true}$

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Variable Elimination

Given a set of polynomials $\mathcal{P} \subset \mathbb{F}_q[x_1, \ldots, x_k]$.

We eliminate x_k by generating set $\mathcal{P}' \subset \mathbb{F}_q[x_1, \ldots, x_{k-1}]$ s.t.

$$(\alpha_1, \ldots, \alpha_{k-1}) \in \mathsf{zero}(\mathcal{P}') \quad \text{iff} \quad \exists \beta \in \mathbb{F}_q. (\alpha_1, \ldots, \alpha_{k-1}, \beta) \in \mathsf{zero}(\mathcal{P})$$

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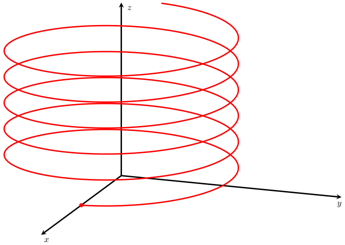

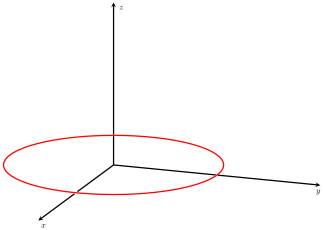

Single incompatibility

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- Assignment $\nu[\Gamma][y \mapsto \alpha_y]$ violates G for all $\alpha_y \in \mathbb{F}_q$

$$(x-1) \cdot y - 1 \qquad\qquad (x-1) \in \mathbb{F}_q[x]$$

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Coefficient based explanation generation

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$$p = c_1 \cdot x_k^{d_1} + \cdots + c_m \cdot x_k^{d_m} \qquad c_i \in \mathbb{F}_q[x_1, \ldots, x_{k-1}]$$

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- Similar for $G := (p \neq 0)$
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Multiple incompatibilities - Gröbner Basis

- Let $\Gamma = [\![\ldots, x_{k-1} \mapsto \alpha_{k-1}, F_1, \ldots F_m]\!]$ and $G_1, \ldots, G_n$ are incompatible
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- Gröbner basis with a lexicographical term ordering has the projection property.
- Introduce fresh variable z for negations

$$f'(x_1, \ldots, x_k, z) = z \cdot f(x_1, \ldots, x_k) - 1$$

- Field polynomials $\mathcal{FP} = \{x_i^q - x_i | x_i \in X\}$ are required.
- Generate the $k-1$ elimination ideal of

$$\langle F_1, \cdots, F_m, G_1, \ldots, G_n \rangle + \langle \mathcal{FP} \rangle$$

Multiple incompatibilities - Exclude factors

- Let $f, g \in \mathbb{F}_q[x_1, \ldots, x_k]$ and α an assignment
- Factor univariate polynomials $\nu[\Gamma](f) \in \mathbb{F}_q[x_k]$ and $\nu[\Gamma](g) \in \mathbb{F}_q[x_k]$
- Exclude common irreducible factors

Subresultant Regular Subchain

- **GCD w.r.t. assignment**
- Let $f, g \in \mathbb{F}_q[x_1, \ldots, x_k]$
- $\mathsf{srs}(f, g, x_k) = h_2, \ldots, h_r$
- $i = \mathsf{lc}(g, x_k)$ and $i_\ell = \mathsf{lc}(h_\ell, x_k)$

$$\gcd(f(\boldsymbol{\alpha}, x_k), g(\boldsymbol{\alpha}, x_k)) = h_\ell(\boldsymbol{\alpha}, x_k)$$

if $\boldsymbol{\alpha} \in \mathsf{zero}(\{i_{\ell+1}, \ldots, i_r\}/\{i, i_\ell\})$

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$\mathbb{F}_q[X]$

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Think of **"Euclidean Division algorithm"** w.r.t. current assignment

Example: SRS

$f = z^2 + yz + 4 = 0$ and $g = x + yz \neq 0 \in \mathbb{F}_5[x, y, z]$

Let $\alpha = \{x \mapsto 3, y \mapsto 1\}$ be the current assignment on Γ

- f and g are incompatible with Γ

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Let $\alpha = \{x \mapsto 3, y \mapsto 1\}$ be the current assignment on Γ

- f and g are incompatible with Γ
- $\mathsf{srs}(f, g, z) = [x + yz, x^2 - xy^2 - y^2]$
- Learn $x^2 - xy^2 - y^2 \neq 0$

In addition to $\{x \mapsto 3, y \mapsto 1\}$ we also exclude $\{x \mapsto 0, y \mapsto 0\}$ and $\{x \mapsto 3, y \mapsto 4\}$

Results

$\mathbb{F}_q[X]$

Type	q	n	c	FFSAT	GB	GBLEX
Craft	3	32	32	**25**	**25**	0
Craft	3	64	64	**25**	24	0
Craft	13	32	16	**19**	18	1
Craft	211	16	8	24	**25**	**25**
Rand	3	8	8	**25**	**25**	**25**
Rand	3	16	16	**12**	11	0
Rand	13	8	4	**25**	0	0
Rand	13	8	8	**1**	0	0
Rand	211	8	4	**17**	0	0
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Instances solved by FFSAT, GB, and GBLEX, out of 25 polynomial systems per test set within 300 seconds.

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$\Rightarrow$ POLYSAT

PolySAT: a Word-level Solver for Large Bitvectors

$\mathbb{Z}/2^k\mathbb{Z}[X]$

Bitvectors?

1. Sequence of bits, e.g., `01011`
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PolySAT: a Word-level Solver for Large Bitvectors

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- $2x^2y + z = 3$
- $x + 3 \leq x + y$
- $\neg\Omega^*(x, y), \quad z = x \,\&\, y, \quad \ldots$

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Natural target for many program verification tasks!

Certora and smart contract verification: 256-bit unsigned integers

Bitvector Pitfalls

$\mathbb{Z}/2^k\mathbb{Z}$ is a finite commutative ring, but not a field.

$x, y \geq 0 \not\Longrightarrow xy \geq x$	Overflow/wraparound: $3 \cdot 6 = 2 \mod 2^4$
$x, y \neq 0 \not\Longrightarrow xy \neq 0$	Zero divisors: $6 \cdot 8 = 0 \mod 2^4$
$x \leq y \not\Longrightarrow x - y \leq 0$	Usual inequality normalization fails

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Example

$x + 3 \leq x + y \mod 2^3$

- For $x = 0$: $3 \leq y \iff y \in \{3, 4, 5, 6, 7\}$
- For $x = 2$: $5 \leq 2 + y \iff y \in \{3, 4, 5\}$

Solving Approaches

- Bit-blasting

  Translate into boolean formula and use SAT solver

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- MCSAT-based approaches **[Zeljić et al., SAT'16]** **[Graham-Lengrand et al., IJCAR'20]**

  Search for assignment to bitvector variables $\rightsquigarrow$ PolySAT

PolySAT Overview

$\mathbb{Z}/2^k\mathbb{Z}[X]$

- Theory solver for bitvector arithmetic
  - Input: conjunction of bitvector constraints
  - Output: SAT or UNSAT
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  - Input: conjunction of bitvector constraints
  - Output: SAT or UNSAT
- Based on modular integer arithmetic ($\mathbb{Z}/2^k\mathbb{Z}$)
- Search for a model of the input constraints
  - Incrementally assign bitvector variables
  - Keep track of viable values for variables
  - Add lemmas on demand to generate explanation clauses

Bitvector Constraints in PolySAT

Inequalities	$p \leq q$	(polynomials p, q)
Overflow	$\Omega^*(p, q)$	
Bit-wise	$r = p \mathbin{\&} q$	
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Quotient/remainder	$q \coloneqq \mathtt{bvudiv}(a, b),\ r \coloneqq \mathtt{bvurem}(a, b)$	
	▶ $a = bq + r$	
	▶ $\neg\Omega^*(b, q)$	
	▶ $\neg\Omega^+(bq, r)$	(e.g., $bq \leq -r - 1$)
	▶ $b \neq 0 \rightarrow r < b$	

PolySAT Solving Loop

Modified CDCL loop with theory assignments

- Assign boolean values to constraint literals ($p \leq q$ vs. $p > q$)
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Main components:

- Trail Γ records assignments and reasons
- For each variable x, keep track of viable values V_x
- Conflict $\mathcal{C}$: set of constraints that contradicts Γ
- Conflict analysis learn a new constraint to avoid the conflict in the future

Example: Polynomial Equations

$\mathbb{Z}/2^k\mathbb{Z}[X]$

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\begin{aligned}
C_1: \quad & x^2y + 3y + 7 = 0 \mod 2^4 \\
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   Conflict: $\mathcal{C} = \{C_3, x = 0, y = 3, z = 2\}$

$\mathbb{Z}/2^k\mathbb{Z}[X]$

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Lemma:

$$C_3 \wedge C_2 \rightarrow 3x + 1 = 0$$

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   Conflict due to parity!

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7. Unsatisfiable.

How to choose values?

For each variable x, keep track of viable values V_x:

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- propagate $x \mapsto v$ when $V_x = \{v\}$ is a singleton set
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Currently:

- V_x represented as set of intervals
- when x appears only linearly, extract a forbidden interval **[Graham-Lengrand et al., IJCAR'20]**
- additionally, keep track of fixed bits of x **[Zeljić et al., SAT'16]**
- bit-blasting as fallback
  (only a single bitvector variable)

Intervals

We use half-open intervals:

- Usual notation $[\ell; u[$
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Examples mod 2^4:

$$[2;5[= \{2,3,4\}$$
$$[13;2[= \{13,14,15,0,1\}$$
$$[0;0[= \emptyset$$

Note:

$$p \in [\ell; u[\quad \iff \quad p - \ell < u - \ell$$

Forbidden Intervals

Forbidden interval of a constraint (example in $\mathbb{Z}/2^4\mathbb{Z}$):

- Current trail Γ contains $x_1 \mapsto 11$, $x_2 \mapsto 13$, and $x_3 \mapsto 9$.

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  $\rightsquigarrow$ use to restrict V_y
- Symbolic interval: $y \notin [-x_1^2 x_3; x_1 - x_1^2 x_3[$

Forbidden Interval Lemma

- Forbidden intervals:
  $C_i \implies x \notin [\ell_i; u_i[$

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$\mathbb{Z}/2^k\mathbb{Z}[X]$

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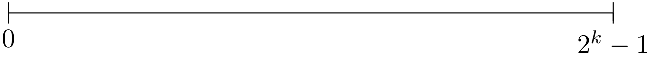

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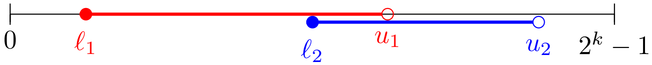

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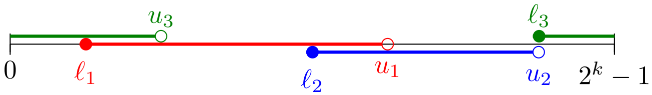

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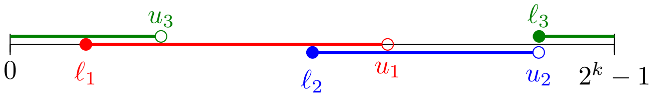

- Use symbolic intervals to express the overlap condition:

$$u_1 \in [\ell_2; u_2[\wedge u_2 \in [\ell_3; u_3[\wedge u_3 \in [\ell_1; u_1[$$

Forbidden Intervals

p, q, r, s: polynomials, evaluable in current trail Γ

x: variable, unassigned in Γ

$$px + r \leq qx + s$$

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[Graham-Lengrand et al., IJCAR'20]

p	q	Interval	
0	1	$x \notin [-s; r - s[$	if $r \neq 0$
1	0	$x \notin [s - r + 1; -r[$	if $s \neq -1$
1	1	$x \notin [-s; -r[$	if $r \neq s$

Forbidden Intervals

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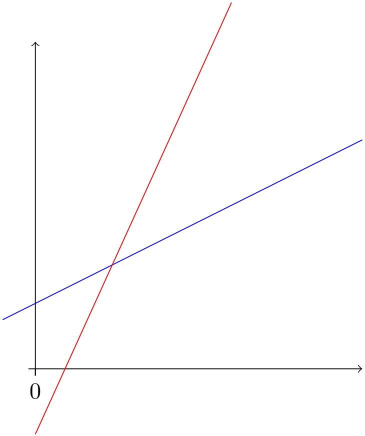
x: variable, unassigned in Γ

$$px + r \leq qx + s$$

p	q	Lemmas from intervals
$\{0, n\}$	$\{0, n\}$	Set of intervals (“equal coeff.”)
n	m	Set of intervals (“disequal coeff.”)
		Intervals from fixed bits
		Fallback to bit-blasting

Forbidden Intervals (disequal coefficients)

$px + r \leq qx + s$ with $p \neq q$



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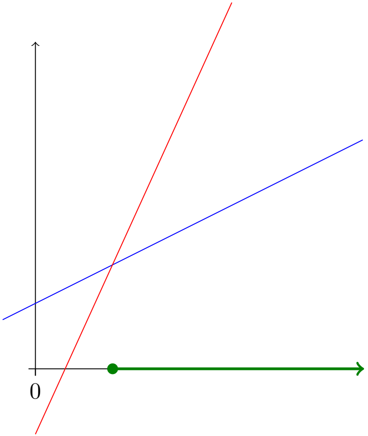

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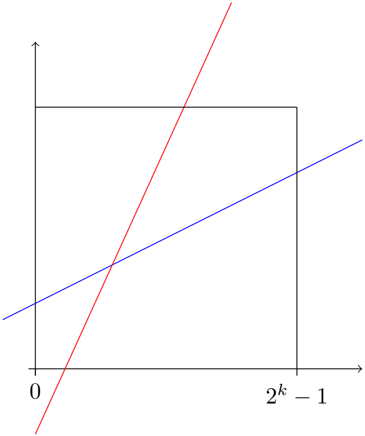

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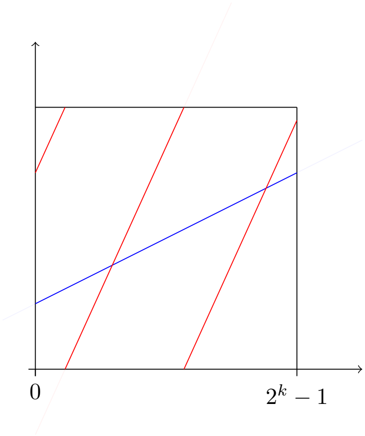

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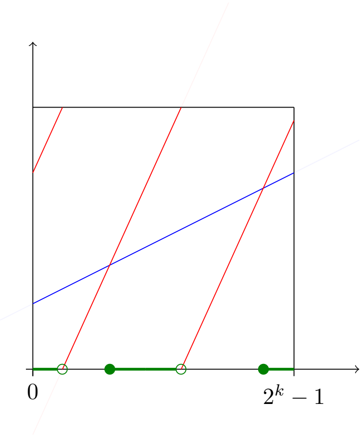

Conflict Resolution Strategy

1. Track the conflict's cone of influence while backtracking over the trail Γ
2. Conflict resolution plugins derive lemmas from constraints in the conflict
3. Accumulate lemmas from conflict plugins
   - New (often simpler) constraints improve propagation
   - Easy to experiment with new types of lemmas
4. When reaching the first relevant decision, learn lemmas and resume search

Conflict Resolution Plugins

Forbidden Intervals Lemma

Conflict Resolution Plugins

Forbidden Intervals Lemma

Superposition	$p(x) = 0 \wedge q(x) = 0$	$\implies rp(x) + sq(x) = 0$
	choose r, s to eliminate highest power of x	

Conflict Resolution Plugins

Forbidden Intervals Lemma		
Superposition	$p(x) = 0 \wedge q(x) = 0$	$\implies rp(x) + sq(x) = 0$
	choose r, s to eliminate highest power of x	
Var. Elim.	$px = q \wedge C[rx + s] \wedge \ \ldots$	$\implies C[p^{-1}q \cdot (r \gg n) + s]$
	pseudo-inverse: $p^{-1}p = 2^n$ for minimal n	

Conflict Resolution Plugins

$\boxed{\mathbb{Z}/2^k\mathbb{Z}[X]}$

Forbidden Intervals Lemma

Superposition	$p(x) = 0 \wedge q(x) = 0$	$\Longrightarrow$	$rp(x) + sq(x) = 0$
	choose r, s to eliminate highest power of x		
Var. Elim.	$px = q \wedge C[rx + s] \wedge \dots$	$\Longrightarrow$	$C[p^{-1}q \cdot (r \gg n) + s]$
	pseudo-inverse: $p^{-1}p = 2^n$ for minimal n		
Bounds	$C(x, y) \wedge x \in [x_l; x_h]$	$\Longrightarrow$	$y \in [y_l; y_h]$
	$\Omega^*(p, q) \wedge p \leq b_1$	$\Longrightarrow$	$q \geq b_2$
	$axy + bx + cy + d \leq \dots$	$\Longrightarrow$	$\dots$

Conflict Resolution Plugins

$\boxed{\mathbb{Z}/2^k\mathbb{Z}[X]}$

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Overflow	$\Omega^*(p, q) \land \neg\Omega^*(p, r)$	$\implies$	$q > r$

Conflict Resolution Plugins

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Overflow	$\Omega^*(p, q) \wedge \neg\Omega^*(p, r)$	$\Longrightarrow$	$q > r$
Bit-wise and	$x = p \,\&\, q$	$\Longrightarrow$	$x \leq p$
	$x = p \,\&\, q \wedge p = q$	$\Longrightarrow$	$x = p$
	$x = p \,\&\, q \wedge p = 2^n - 1$	$\Longrightarrow$	$2^{n-k}x = 2^{n-k}q$
...	...	...	

MCSAT based approaches for non-linear modular arithmetic

1. Constraints in $\mathbb{F}_q[X]$

- Finite field
- Not algebraically closed
- Constraints: $=, \neq$

Modulo 5

$$
\begin{aligned}
x^2 - 1 &= 0 \\
xy - y - 1 &= 0 \\
xy - 2 &\neq 0
\end{aligned}
$$

$\Rightarrow$ FFSat

2. Constraints in $\mathbb{Z}/2^k\mathbb{Z}[X]$

- Finite commutative ring
- Not algebraically closed
- Constraints: $=, \neq, <, >, \Omega^*(x, y)$

Modulo 2^4

$$
\begin{aligned}
xy + y &\leq y + 3 \\
2y + z &= 10 \\
3x + 6yz + 3z^2 &= 1
\end{aligned}
$$

$\Rightarrow$ PolySat