Certified Static and Dynamic Symmetry Breaking

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(Thanks to co-conspirators Jo Devriendt, Ward Gauderis,
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Vrije Universiteit Brussel

Satisfiability: Theory, Practice, and Beyond
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ARTIFICIAL
INTELLIGENCE
RESEARCH GROUP

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  - σ respects negation: $\overline{\sigma(x)} = \sigma(\overline{x})$
  - $F\restriction_\sigma = F$
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Example

Consider the formula F:

$$\begin{array}{ll} a \lor \overline{b} \lor x \lor y & b \lor c \lor x \lor y \\ \overline{c} \lor \overline{d} \lor x \lor y & d \lor \overline{a} \lor x \lor y \end{array}$$

The permutation

$$(ab\overline{c}d)(xy)(\overline{a}\overline{b}c\overline{d})(\overline{x}\overline{y})$$

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  - $F\upharpoonright_\sigma = F$
    ($F\upharpoonright_\sigma$ is replacing each x by $\sigma(x)$ in F)
- Symmetric problems are often problematic for vanilla CDCL solvers (insert obligatory reference to PH principle here)

Example

Consider the formula F:

$$a \vee \overline{b} \vee x \vee y \qquad b \vee c \vee x \vee y$$

$$\overline{c} \vee \overline{d} \vee x \vee y \qquad d \vee \overline{a} \vee x \vee y$$

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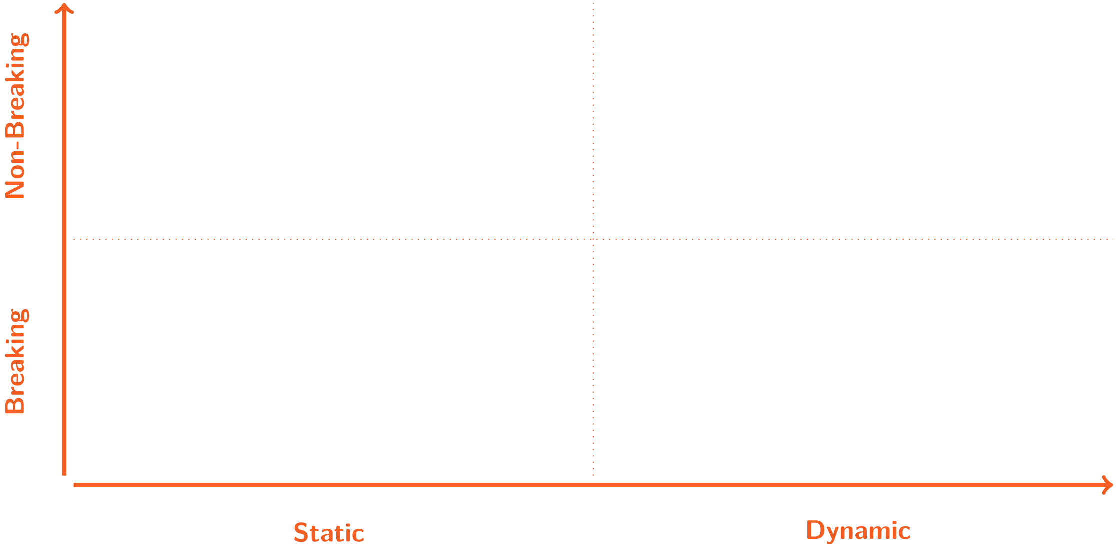
OUTLINE OF THIS TALK

1. Introduction
2. Handling Symmetries in SAT (Overview)
3. Symmetry Breaking with VeriPB
    1. The VeriPB proof System
    2. VeriPB-certified symmetry breaking
4. Conclusion

SYMMETRY HANDLING TECHNIQUES



Non-Breaking

Breaking

Static

Dynamic

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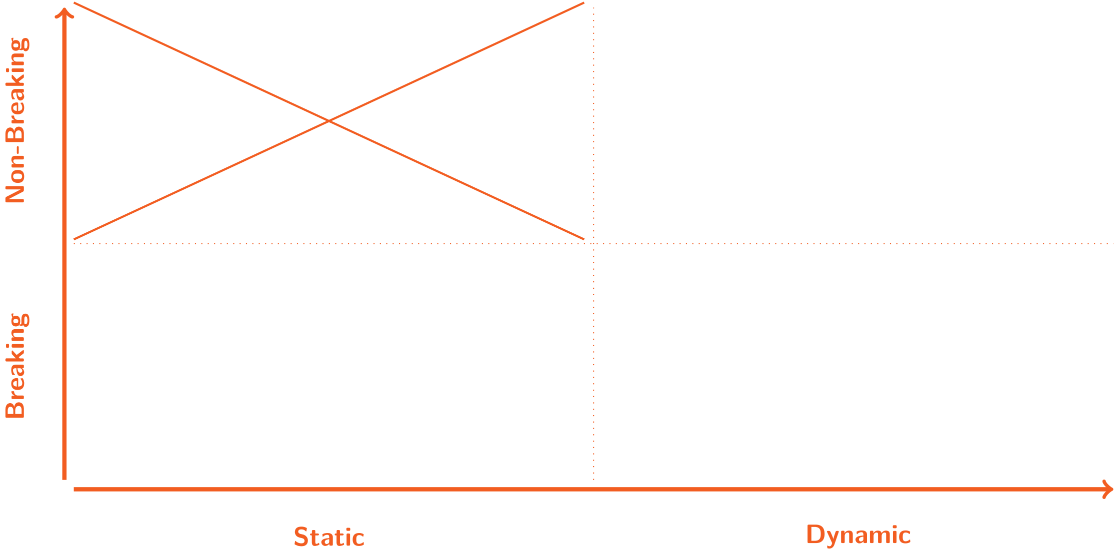

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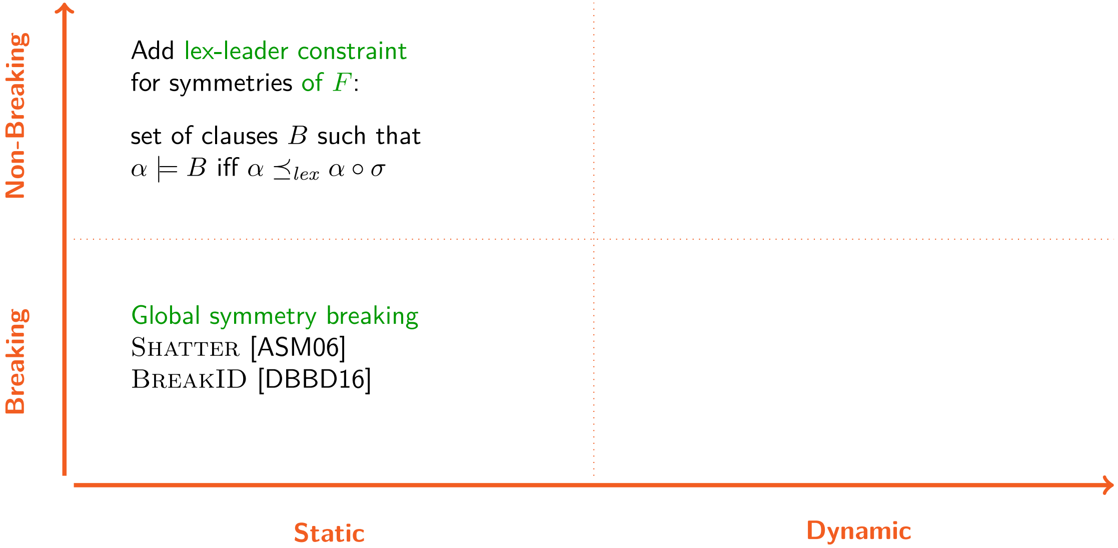

Add lex-leader constraint
for symmetries of F:

set of clauses B such that
$\alpha \models B$ iff $\alpha \preceq_{lex} \alpha \circ \sigma$

Global symmetry breaking
Shatter [ASM06]
BreakID [DBBD16]

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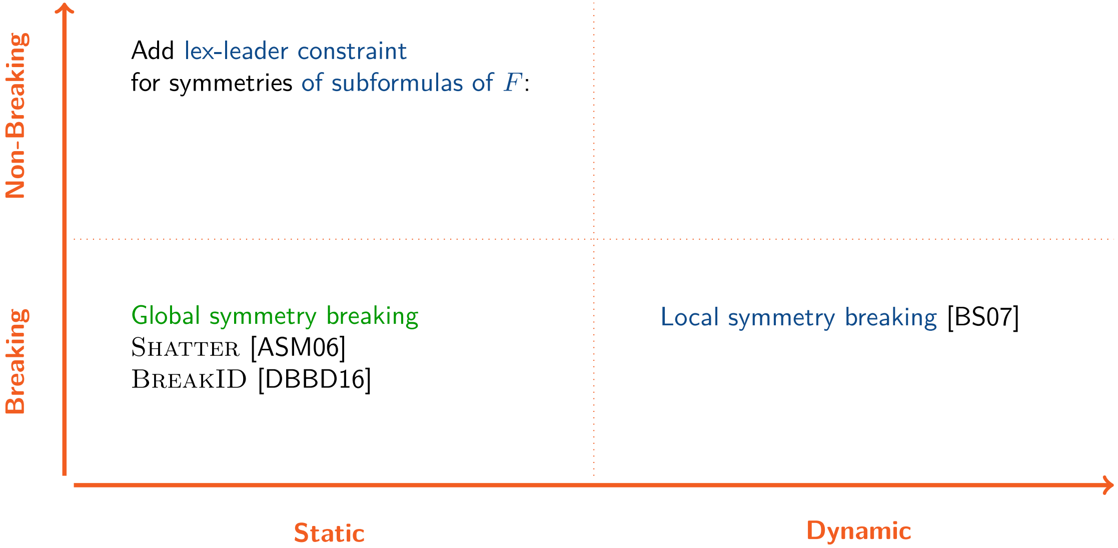

Add lex-leader constraint
for symmetries of subformulas of F:

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Local symmetry breaking [BS07]

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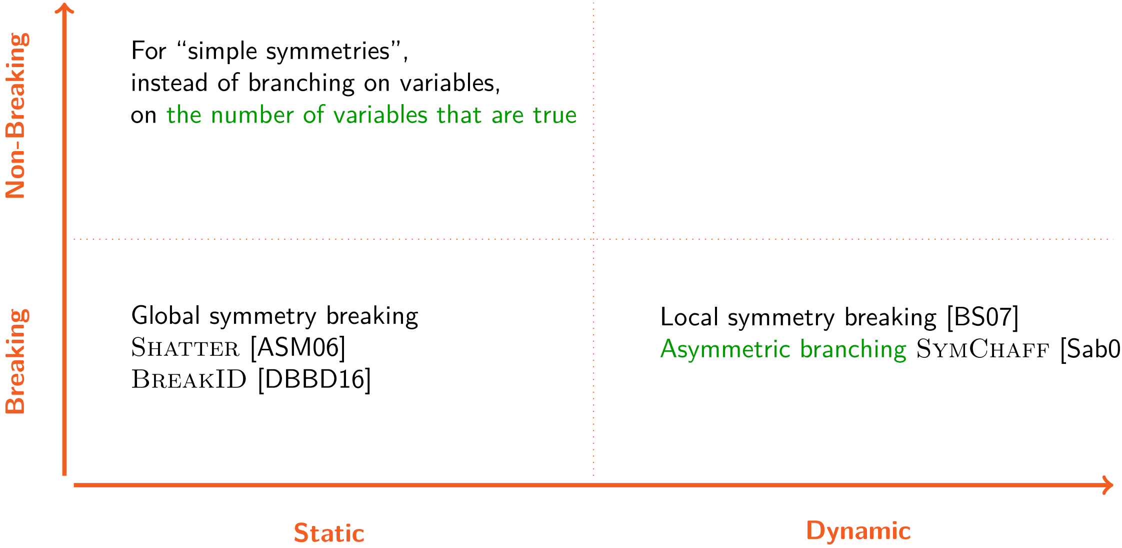

For "simple symmetries",
instead of branching on variables,
on the number of variables that are true

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Local symmetry breaking [BS07]
Asymmetric branching SymChaff [Sab09]

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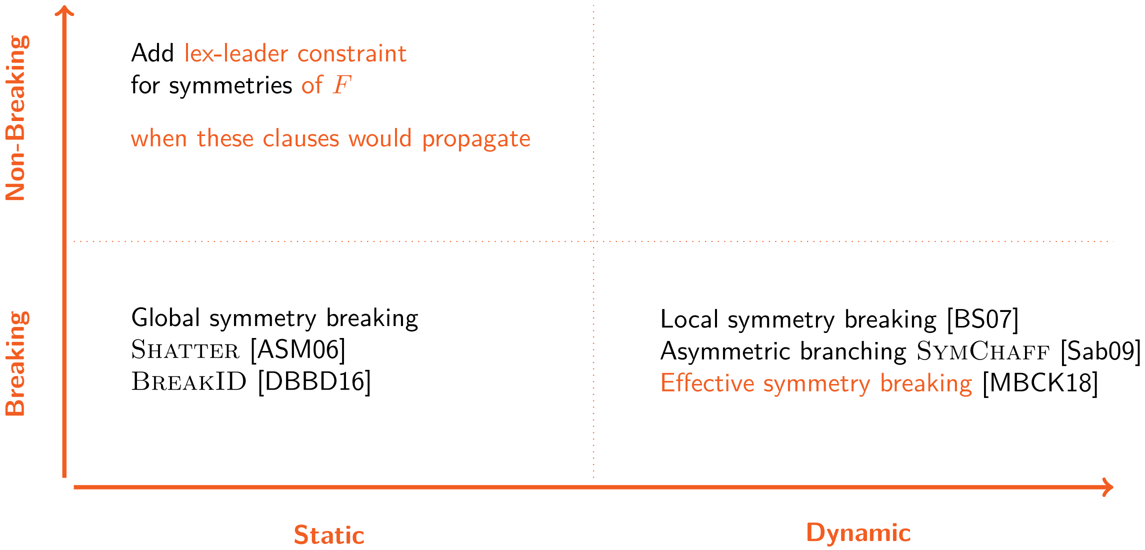

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Add lex-leader constraint
for symmetries of F

when these clauses would propagate

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Asymmetric branching SymChaff [Sab09]
Effective symmetry breaking [MBCK18]

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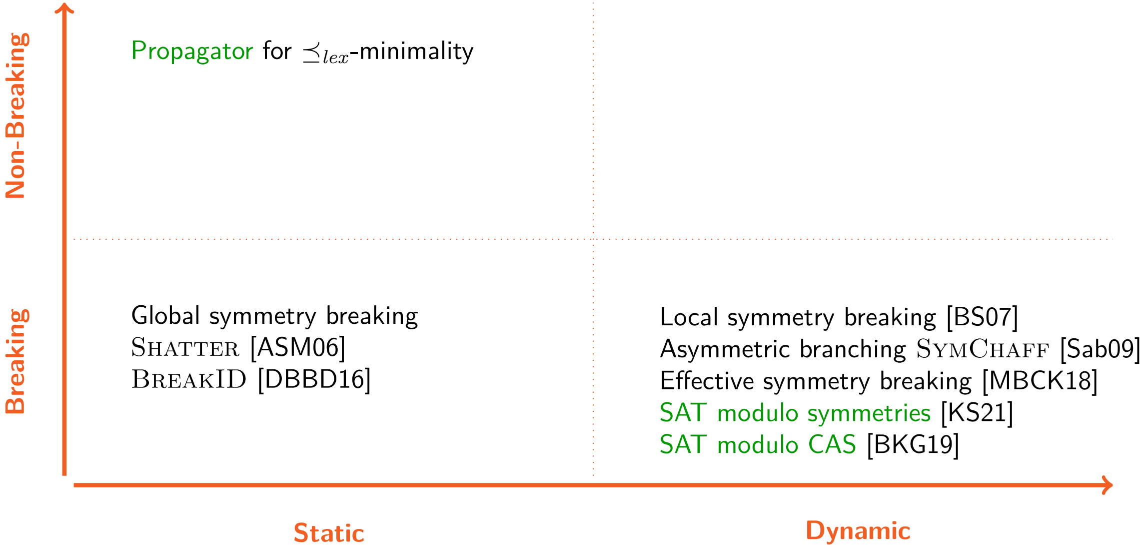

Propagator for $\preceq_{lex}$-minimality

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SAT modulo symmetries [KS21]
SAT modulo CAS [BKG19]

SYMMETRY HANDLING TECHNIQUES

	Static	Dynamic
Non-Breaking	When SAT solver learns c, also learn $c{\restriction}_{\sigma}$ (if this seems “interesting”)	Symmetric Learning [HKM^{+}05] [SHvM09, BNOS10, DBD^{+}12, DBB17]
Breaking	Global symmetry breaking Shatter [ASM06] BreakID [DBBD16]	Local symmetry breaking [BS07] Asymmetric branching SymChaff [Sab09] Effective symmetry breaking [MBCK18] SAT modulo symmetries [KS21] SAT modulo CAS [BKG19]

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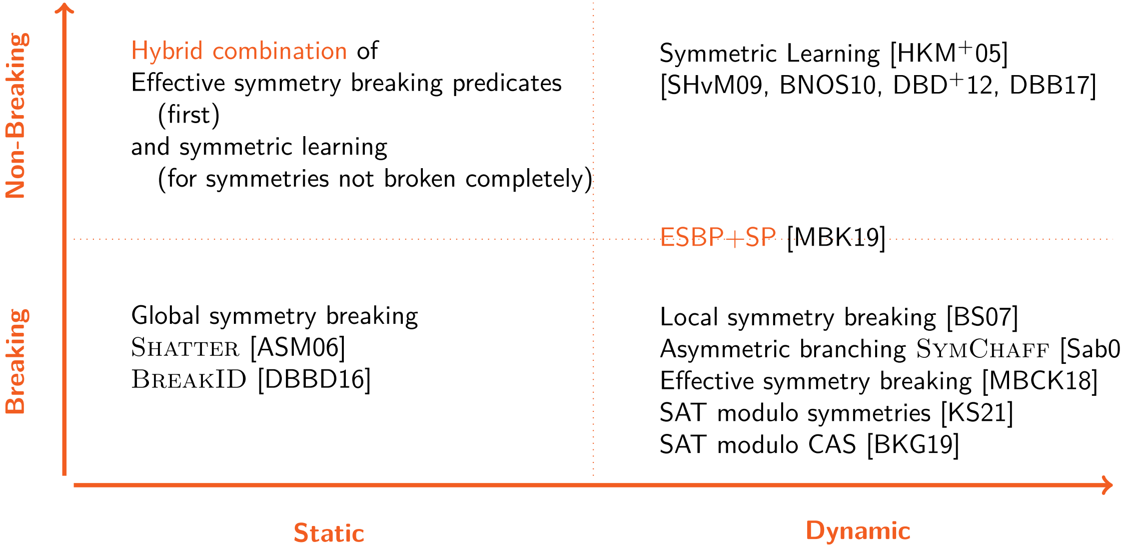

Non-Breaking

Hybrid combination of
Effective symmetry breaking predicates
(first)
and symmetric learning
(for symmetries not broken completely)

Symmetric Learning [HKM$^+$05]
[SHvM09, BNOS10, DBD$^+$12, DBB17]

ESBP+SP [MBK19]

Breaking

Global symmetry breaking
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  - working version of ESBP [MBCK18]
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  preliminary results
  - working version of ESBP [MBCK18]
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Symmetric learning

- Recently proposed proof logging [TD20]
  1. Special-purpose, specific approach
  2. Requires adding explicit concept of symmetries
  3. Not compatible with preprocessing techniques

  Better to keep proof system super-simple(?)

THE VERIPB PROOF SYSTEM

A proof system for **pseudo-Boolean optimization problems**

- Reasons with general pseudo-Boolean constraints
- Builds on cutting planes
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Details about the proof checker, see Stephan Gocht's PhD thesis [Goc22]

PSEUDO-BOOLEAN CONSTRAINTS

Pseudo-Boolean constraints are 0-1 integer linear constraints

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- as before, variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

PSEUDO-BOOLEAN REASONING: CUTTING PLANES [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

REDUNDANCE-BASED STRENGTHENING

- C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable
- Adding redundant constraints should be OK
- Notions such as RAT [JHB12] and propagation redundancy [HKB17]

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Redundance-based strengthening [BT19, GN21]

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$$

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$
- Implication should be efficiently verifiable (which is the case, e.g., if all constraints in $(F \wedge C)\restriction_\omega$ are RUP)

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Deal with symmetries by switching focus to optimization

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Spoiler alert:
For decision problem, nothing stops us from inventing objective function
(like lexicographic order $\sum_{i=1}^{n} 2^i \cdot x_i$)

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7. . . .
8. Can't go on forever, so finally reach α' satisfying $F \wedge D$

STRENGTH OF DOMINANCE RULE

Dominance-based strengthening (stronger, still simplified) [BGMN22]

If $D_1, D_2, \ldots, D_{m-1}$ have been derived from F (maybe using dominance), then can derive also D_m if exists witness substitution ω such that

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Why is this sound?

- Same inductive proof as before, but nested

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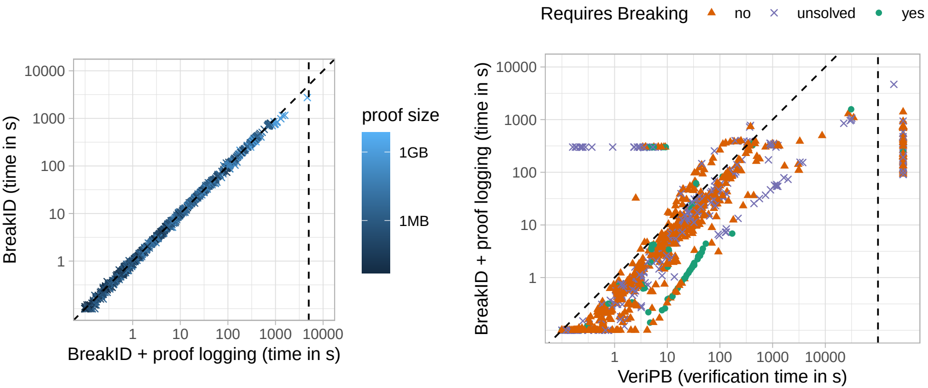
$$\begin{aligned} &y_0 && \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j \\ &\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) && y_j \vee \overline{y}_{j-1} \vee \overline{x}_j \\ &\overline{y}_j \vee y_{j-1} && y_j \vee \overline{y}_{j-1} \vee \sigma(x_j) \end{aligned}$$

EXPERIMENTAL EVALUATION

- Evaluated on SAT competition benchmarks
- BREAKID [DBBD16, Bre] used to find and break symmetries



- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances

BREAKING SYMMETRIES WITH THE DOMINANCE RULE (1/2)

Definition

Given a symmetry σ, the (pseudo-Boolean) breaking constraint of σ is

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Theorem

C_σ can be derived from F using dominance with witness σ

$$F \wedge \neg C_\sigma \models F\upharpoonright_\sigma \wedge f\upharpoonright_\sigma < f$$

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Breaking symmetries with the dominance rule

- Surprisingly simple

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Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

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Thank you for your attention!

SYMMETRY BREAKING: EXAMPLE

Example (Pigeonhole principle formula)

- Variables p_{ij} $(1 \leq i \leq 4, 1 \leq j \leq 3)$ true iff pigeon i in hole j
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Order: "Pigeon 1 preferred in the largest possible hole; next pigeon 2, ..."

$$f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^{9} \cdot p_{11} + 2^{8} \cdot p_{23} + \cdots + 1 \cdot p_{41}$$

BREAKING A SINGLE SIMPLE SYMMETRY (EXAMPLE)

- F is a formula expressing PHP constraints with $F{\upharpoonright}_{\sigma_{(12)}} = F$
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Similar to DRAT symmetry breaking [HHW15]

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Problem

This idea does not generalize.

- Breaking two symmetries

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Intuitively, C_{1234} holds if shifting all the pigeons results in a worse assignment.
If it is falsified, we can “restore” its truth by applying $\sigma_{(1234)}$ once, twice, or thrice.

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1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (searching lexicographically smallest assignment satisfying formula)
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Define y_j to be true if x_k equals $\sigma(x_k)$ for all $k \leq j$

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(derivable with redundance rule)

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(derivable with redundance rule) If y_k is true, x_k is at most $\sigma(x_k)$
(derivable from the PB breaking constraint)

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

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Pseudo-Boolean breaking constraint

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y_0

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$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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with

$$\sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \leq 2^{n-1} - 1$$

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$\overline{y}_1 \vee y_0$

Derivable by redundance with witness $\omega : y_1 \mapsto 0$

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$$\models (F \wedge D)\restriction_\omega \wedge \{\overline{y}_1 \vee y_0\}\restriction_\omega$$

$$F \wedge D \wedge \neg(\overline{y}_1 \vee y_0)$$

$$\models (F \wedge D)\restriction_\omega \wedge \{1 \vee y_0\}$$

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$\overline{y}_1 \vee y_0$

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Derivable by redundance with witness $\omega : y_1 \mapsto 0$ (same argument)

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$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

$\overline{y}_1 \vee y_0$

$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$

$y_1 \vee \overline{y}_0 \vee \overline{x}_1$

Derivable by redundance with witness $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 \vee \overline{y}_0 \vee \overline{x}_1) \models (F \wedge D)\restriction_\omega \wedge \{y_1 \vee \overline{y}_0 \vee \overline{x}_1\}\restriction_\omega$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

Derivable by redundance with witness $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 \vee \overline{y}_0 \vee \overline{x}_1)$$
$$\models (F \wedge D)\restriction_\omega \wedge \{y_1 \vee \overline{y}_0 \vee \overline{x}_1\}\restriction_\omega$$
$$F \wedge D \wedge \{\overline{y}_1 \wedge y_0 \wedge x_1)$$
$$\models \cdots \wedge D\restriction_\omega \wedge \dots$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

Derivable by redundance with witness $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 \vee \overline{y}_0 \vee \overline{x}_1)$$

$$\models (F \wedge D)\restriction_\omega \wedge \{y_1 \vee \overline{y}_0 \vee \overline{x}_1\}\restriction_\omega$$

$$F \wedge D \wedge \{\overline{y}_1 \wedge y_0 \wedge x_1)$$

$$\models \dots \wedge D\restriction_\omega \wedge \dots$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

Derivable by redundance with witness $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 \vee \overline{y}_0 \vee \overline{x}_1)$$
$$\models (F \wedge D)\restriction_\omega \wedge \{y_1 \vee \overline{y}_0 \vee \overline{x}_1\}\restriction_\omega$$
$$F \wedge D \wedge \{\overline{y}_1 \wedge y_0 \wedge x_1)$$
$$\models \cdots \wedge D\restriction_\omega \wedge \ldots$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

$$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$$

Derivable by redundance with witness $\omega : y_1 \mapsto 1$ (same argument)

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

$\overline{y}_1 \vee y_0$

$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$

$y_1 \vee \overline{y}_0 \vee \overline{x}_1$

$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$

$\overline{y}_1 \vee \overline{x}_2 \vee \sigma(x_2)$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

$\overline{y}_1 \vee y_0$

$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$

$y_1 \vee \overline{y}_0 \vee \overline{x}_1$

$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$

$\overline{y}_1 \vee \overline{x}_2 \vee \sigma(x_2)$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$+ 2^{n-1} \cdot \left(\overline{y}_1 + \overline{\sigma(x_1)} + x_1 \geq 1\right)$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \lor \overline{x}_1 \lor \sigma(x_1)$$

$$\overline{y}_1 \lor y_0$$

$$\overline{y}_1 \lor \overline{\sigma(x_1)} \lor x_1$$

$$y_1 \lor \overline{y}_0 \lor \overline{x}_1$$

$$y_1 \lor \overline{y}_0 \lor \sigma(x_1)$$

$$\overline{y}_1 \lor \overline{x}_2 \lor \sigma(x_2)$$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$+\, 2^{n-1} \cdot \left(\overline{y}_1 + \overline{\sigma(x_1)} + x_1 \geq 1\right)$$

$$2^{n-1} \cdot \overline{y}_1 + \sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

$$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee \overline{x}_2 \vee \sigma(x_2)$$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$+ 2^{n-1} \cdot \left(\overline{y}_1 + \overline{\sigma(x_1)} + x_1 \geq 1\right)$$

$$2^{n-1} \cdot \overline{y}_1 + \sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

The clause to derive is RUP wrt this constraint

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