A Combinatorial Characterization of Minimax in 0/1 Games

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Background

- Last year we were trying to prove a statement in *Online Learning*
- We found a proof which is based on the *Minimax Theorem*

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- Consequently, our proof did not apply in the general case
- not uncommon in ML theory: usually “swept under the rug”

Background

is learnable if and only if its Ψ-dimension is finite. After Vapnik [18], we will adopt a naive attitude toward measurability, assuming that every set encountered in our proofs is measurable. If one prefers, one may assume that the domain of any probability space we describe is countable, although considerably weaker assumptions, similar to those used in [4, 7], suffice. If X is a set, P is a probability distribution over X, and f maps X to $\mathbf{R}$, let $\mathbf{E}_{x \in P}[f(x)]$ denote the expectation of f with respect to P.

Background

An assumption. Some of our arguments exploit the Minimax Theorem for zero-sum games [von Neumann, 1928]. Therefore, we will assume a setting (i.e. the domain $\mathcal{X}$ and the set of distributions $\mathcal{Q} \subseteq \Delta(\mathcal{X})$) in which this theorem is valid. Alternatively, one could state explicit assumptions such as finiteness or forms of compactness under which it is known that the Minimax Theorem holds. However, we believe that the presentation benefits from avoiding such explicit technical assumptions and simply assuming the Minimax Theorem as an "axiom" in the discussed setting.

Background

- It sometimes makes sense to lose generality in order to simplify presentation and focus on key (and novel) ideas
- However, the result we'll discuss was discovered while attempting to generalize the result beyond what was "needed"
- Curiously, it seems this side-result turned out to be more popular than the main result (even among ML people)

The Minimax Theorem

John Von-Neumann: *“As far as I can see, there could be no theory of games… without that theorem… I thought there was nothing worth publishing until the Minimax Theorem was proved”*

The Minimax Theorem

Definitions.

- Two players: Minnie and Max
- S – Minnie's Pure Strategies
- T – Max's Pure Strategies
- $M: S \times T \rightarrow \{0,1\}$ – The Payoff Matrix
    - $M(s,t) = 1 \rightarrow$ Max wins
    - $M(s,t) = 0 \rightarrow$ Minnie wins

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Minnie and Max may use randomized (aka mixed) strategies:

- p – a distribution over S
- q – a distribution over T

- $M(p,q) := \mathop{\mathbf{E}}_{s \sim p, t \sim q}[M(s,t)]$ = the prob that Max wins/Minnie loses

The Minimax Theorem

Theorem. [von Neumann 1928]
If S and T are finite then

$$\min_p \max_q M(p,q) = \max_q \min_p M(p,q)$$

- $\min_p \max_q M(p,q)$:
  First Minnie picks p and then Max picks $q = q(p)$ – Max has advantage
- $\max_q \min_p M(p,q)$
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Minimax: it doesn't matter who plays first!
If $\forall$ strategy of oponnent $\exists$ a **response** that wins it w.p $\geq v$
Then $\exists$ **univeral strategy** that wins $\forall$ oponnent's strategy w.p $\geq v$

The Minimax Theorem

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If S and T are finite then
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Definition. [Game Value]
Let M be a game which satisfies the Minimax Theorem. Define:
$$\mathrm{val}(M) := \min_p \max_q M(p,q) = \max_q \min_p M(p,q)$$

Minimax for Infinite Games?

Question. Does The Minimax Theorem Apply to Infinite Games?

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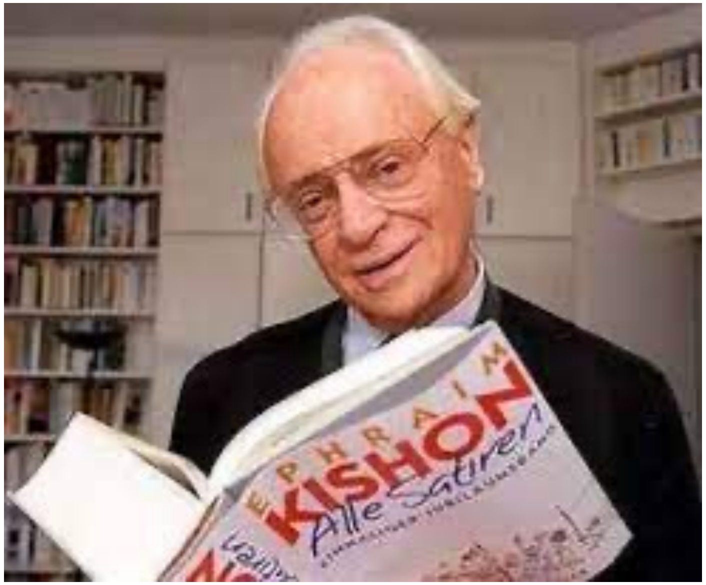
There are simple infinite games for which the Minimax fails to hold.

Minimax for Infinite Games?

We will now see such a game popularized by Ephraim Kishon that does not satisfy minimax



Ephraim
Kishon
1924-2005

Kishon’s Jewish Poker

For quite a while the two of us sat at our table, wordlessly stirring our coffee. Ervinke was bored.
”*All right”,* he said.
”*Let's play poker”*.

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“No”, I answered.
”*I hate cards. I always lose.”*
“Who's talking about cards? ” thus Ervinke.
“I was thinking of Jewish poker.”
He then briefly explained the rules of the game.
“Jewish poker is played without cards, in your head, as befits the People of the Book.”

Kishon’s Jewish Poker

“*You think of a number, I also think of a number”*, Ervinke said.
“Whoever thinks of a higher number wins.
This sounds easy, but it has a hundred pitfalls. Nu!”
“*All right”*, I agreed.
“Let's try.”
We plunked down five piasters each, and, leaning back in our chairs began to think of numbers. After a while Ervinke signaled that he had one. I said I was ready.

Kishon’s Jewish Poker

“*All right*”, thus Ervinke.
“*Let's hear your number”.*
“*Eleven*”, I said.
“*Twelve*”, Ervinke said, and took the money.
I could have' kicked myself, because originally I had thought of Fourteen, and only at the last moment had I climbed down to Eleven, I really don't know why.

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“*Listen*”. I turned to Ervinke.
“*What would have happened had I said Fourteen?*”
“*What a question! I'd have lost. Now, that is just the charm of poker: you never know how things will turn out. But if your nerves cannot stand a little gambling, perhaps we had better call it off.*”

Kishon’s Jewish Poker

Without saying another word, I put down ten piasters on the table. Ervinke did likewise. I pondered my number carefully and opened with Eighteen.

“Damn!” Ervinke said.

“*I have only Seventeen!”*

I swept the money into my pocket and quietly guffawed.

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“Damn!” Ervinke said.
“*I have only Seventeen!”*
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Ervinke had certainly not dreamed that I would master the tricks of Jewish poker so quickly. He had probably counted on my opening with Fifteen or Sixteen, but certainly not with Eighteen. Ervinke, his brow in angry furrows, proposed to double the stakes.
As you like, I sneered, and could hardly keep back my jubilant laughter. In the meantime a fantastic number had occurred to me: Thirty-five!

Kishon’s Jewish Poker

Lead! said Ervinke.

“Thirty-five!|

“*Forty-three!”*

With that he pocketed the forty piasters.

I could feel the blood rushing into my brain….

Kishon’s Jewish Poker

Let’s present this game in the language of game theory:

- $S = T = \mathbb{N}$

- $M(s, t) = \begin{cases} 1 & s < t \\ 0 & t < s \end{cases}$

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- Payoff Matrix is infinite triangular
- * = can be defined arbitrarily

	0	1	2	$\cdots$			
0	*	1	1	1	1	1	1
1	0	*	1	1	1	1	1
2	0	0	*	1	1	1	1
$\vdots$	0	0	0	*	1	1	1
	0	0	0	0	*	1	1
	0	0	0	0	0	*	1
	0	0	0	0	0	0	*

Jewish Poker Does Not Satisfy Minimax

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Observation. Jewish Poker does not satisfy the Minimax theorem:

$$\inf_{p} \sup_{q} M(p,q) = 1 \neq 0 = \sup_{q} \inf_{p} M(p,q)$$

Main Result: Jewish Poker is The Only Obstacle for Minimax

Theorem. [Hanneke-Livni-M 2021]
Let M be a (possibly infinite) game. Then, if the payoff matrix does not contain arbitrarily large triangular submatrices then

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- "Contains a submatrix" = up to a permutation of rows and columns
- Contra-positively: if the Minimax Theorem fails to hold then there exist arbitrarily large sub-games of jewish poker

Open Questions

Conjecture.
Let M be a (possibly infinite) game. Then, if the payoff matrix does not contain an infinite triangular submatrix then

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Open Questions

Conjecture.
Let M be a (possibly infinite) game. Then, if the payoff matrix does not contain an **infinite** triangular submatrix then

$$\inf_{p} \sup_{q} M(p,q) = \sup_{q} \inf_{p} M(p,q)$$

- Contra-positively: if the Minimax Theorem fails to hold then there jewish poker is a subgame of M

Open Questions

Question.
Is there a “finite manifestation” of this theorem?

- Is there a sense in which finite games with no large triangular submatrices satisfy a more “efficient” version of the Minimax?

- Perhaps Computationally?

Proof Sketch

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Proof.
Assume towards contradiction that there is a game M s.t.

1. M does not satisfy the Minimax, and
2. the largest triangular submatrix of M has bounded size.

We will reach a contradiction to Item 2 by showing that M contains an **infinite** triangular submatrix.

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Lemma 1.

There exist sequences of sets of strategies $S_1, S_2, \ldots \subseteq S$ and $T_1, T_2, \ldots \subseteq T$ s.t.:

- For all i, $|S_i|, |T_i| \leq d$ for some finite d.
- $\mathrm{val}(S_i, T_j) = \begin{cases} < r & i < j, \\ > r & i > j, \end{cases}$ for some $r \in (0,1)$.

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- Proof uses *Uniform Law of Large Numbers* (Vapnik-Chervonenkis '69)

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- Proof uses *Ramsey Theorem* (Ramsey '28) \\same year like Minimax

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Proof.
By assumption:

$$\inf_p \sup_q M(p, q) = \beta > \alpha = \sup_q \inf_p M(p, q)$$

Set $r = \frac{\alpha+\beta}{2}$

We prove the claim by induction.

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- Consider the game where Max's pure strategies are restricted to $\bigcup_{i=1}^{n} T_i$

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- This game satisfies Minimax and has value

$$\mathrm{Val}(\mathrm{S}, \cup_{i=1}^{n} T_i) \leq \alpha$$

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- Hence $\exists p$ such that $\forall b \in \cup_{i=1}^{n} T_i$: $M(p, b) \leq \alpha$

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S_{n+1} is derived by sampling strategies from p

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S_{n+1} is derived by sampling $d = O\left(\frac{\mathrm{VC}(M)}{(\beta-\alpha)^2}\right)$ strategies from p

- Bound on number of samples follows from Vapnik-Chervonenkis Theory
- Payoff matrix has bounded VC dimension

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T_{n+1} is derived by the same argument

■

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Proof.
Consider the following edge-coloring of the complete graph over $\mathbb{N}$:

$$(\forall i < j)\colon \mathrm{color}(\{i,j\}) = \left(\mathrm{payoff}(S_i, T_j), \mathrm{payoff}(S_j, T_i)\right)$$

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By Ramsey, there is infinite monochromatic $\{k_1, k_2, \dots\} \subseteq \mathbb{N}$

- Let (M', M'') denote the color of all edges in $\{k_1, k_2, \dots\}$

Proof Sketch

Lemma 2.
Let $S_1, S_2, \ldots$ and $T_1, T_2, \ldots$ as in Lemma 1.
Then, there is a way to choose $s_i \in S_i$ and $t_j \in T_j$ and a subsequence $k_1, k_2, \ldots$ such that the matrix

$$M\left(s_{k_i}, t_{k_j}\right) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$$

is infinite upper-triangular. \\like jewish poker

$$(\forall i < j): \mathrm{color}(\{i,j\}) = \left(\mathrm{payoff}\left(S_i, T_j\right), \mathrm{payoff}\left(S_j, T_i\right)\right)$$

By Ramsey, there is infinite monochromatic $\{k_1, k_2, \ldots\} \subseteq \mathbb{N}$

- Let (M', M'') denote the color of all edges in $\{k_1, k_2, \ldots\}$
- $\mathrm{val}(M') < r < \mathrm{val}(M'')$ and so there are $a, b \leq d$ s.t:

$$M'(a,b) = 0 < 1 = M''(a,b)$$

■

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- Set s_i to be the a'th strategy in A_{k_i}
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∎

Thank you!