The Analysis of Partially Symmetric Functions

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Based on joint work with

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Classes of “simple” functions
Classes of “simple” functions

Constant
Classes of “simple” functions

Constant

Juntas
Classes of “simple” functions

Constant → Symmetric

Juntas
Classes of “simple” functions

Constant  →  Symmetric

Juntas  →  Partially symmetric
Classes of “simple” functions

Constant $\rightarrow$ Symmetric

Juntas $\rightarrow$ Partially symmetric

**Def’n.** $f : \{0,1\}^n \rightarrow \{0,1\}$ is \((n-k)\)-symmetric if there is a set $J \subseteq [n]$ of $k$ variables such that $f(x) = f(y)$ whenever $x_J = y_J$ and $|x| = |y|$.
An algebraic definition

- Def'n. $f^\pi(x) = f(\pi x) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

- Def'n. $f$ is poly-symmetric if

$$|\text{ISO}_f| = |\{f^\pi : \pi \in S_n\}| \leq \text{poly}(n).$$

- Theorem. $f$ is poly-symmetric if and only if it is $(n-k)$-symmetric for some $k=O(1)$.

[Clote, Kranakis ’91]
[Chakraborty, Fischer, Garcia-Soriano, Matslieh ’12]
Partial Symmetry in Theoretical Computer Science
Circuit complexity

**Theorem** (Shannon ’49). Almost every function $f$ has circuit complexity $\Omega(2^n/n)$. 
Circuit complexity

- **Theorem.** Every symmetric function has circuit complexity at most $n^2$.  
  [Shannon ’38]

- **Theorem.** Every $k$-junta has circuit complexity at most $2^{k+3}/k$.  
  [Shannon ’49]
Circuit complexity

- **Theorem.** Every symmetric function has circuit complexity at most $n^2$. [Shannon ’38]

- **Theorem.** Every $k$-junta has circuit complexity at most $2^{k+3}/k$. [Shannon ’49]

- **Theorem.** Every $(n-k)$-symmetric function has circuit complexity $\leq (n-k)2^k + (n-k)^2$. [Shannon ’49]
Parallel complexity and Proof complexity

- **Theorem.** If $f$ is $(n-k)$-symmetric for some $k=O(1)$, then $f$ is in $\text{TC}^0 \subseteq \text{NC}^1$.

  [Clote, Kranakis ’91]

- **Corollary.** “Frege probably does not effectively-$p$ simulate Extended Frege.”

  [Pitassi, Santhanam ’10]
Testing function isomorphism

\[ f(x) = g(x) \]

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<thead>
<tr>
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<th>( f(x) )</th>
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Testing function isomorphism

**Def’n.** A \textit{q-query tester} for the property $\text{ISO}_f = \{f^\pi : \pi \in S_n\}$ queries $g : \{0,1\}^n \to \{0,1\}$ on at most $q$ inputs and

(i) Accepts w.p. $2/3$ when $g \in \text{ISO}_f$,

(ii) Rejects w.p. $2/3$ when for every $\pi \in S_n$,

$$\Pr[ g(x) \neq f^\pi(x) ] \geq 1/100.$$ 

**Main Question.** For which functions $f$ can we test $\text{ISO}_f$ with $O(1)$ queries?
Testing function isomorphism

- **Fact.** For every symmetric function \( f \), we can test \( ISO_f \) with \( O(1) \) queries.

- **Theorem.** For every \( k \)-junta \( f \), we can test \( ISO_f \) with \( O(k \log k) \) queries.

  [Fischer, Kindler, Ron, Safra, Samorodnitsky ’04]
  [B. ’09]
  [Chakraborty, Garcia-Soriano, Matsliah ’10]
Testing function isomorphism

- **Fact.** For every symmetric function $f$, we can test $\text{ISO}_f$ with $O(1)$ queries.

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  - [B. ’09]
  - [Chakraborty, Garcia-Soriano, Matslieh ’10]

- **Theorem.** For every $(n-k)$-symmetric $f$, we can test $\text{ISO}_f$ with $O(k \log k)$ queries.
  - [B., Weinstein, Yoshida ’12]
  - [Chakraborty, Fischer, Garcia-Soriano, Matslieh ’12]
Conjecture. Fix any $k \geq 1$. If $f$ is $\varepsilon$-far from $(n-k)$-symmetric, then testing $\text{ISO}_f$ requires $\Omega(\log \log k)$ queries.

[B., Weinstein, Yoshida ’12]
[Chakraborty, Fischer, Garcia-Soriano, Matslieh ’12]
Influence and partial symmetry
Three Notions of Influence

Influence of coordinate $i$:

\[ \text{Inf}_i(f) = \Pr_x[ f(x) \neq f(x^{\oplus i}) ] . \]

Total influence / average sensitivity:

\[ \text{Inf}(f) = \sum_i \text{Inf}_i(f) . \]

Influence of a set $S \subseteq [n]$ of coordinates:

\[ \text{Inf}_S(f) = \Pr_{x,y}[ f(x) \neq f(x_{[n]\setminus S}y_S) ] . \]
Three Notions of Influence

Influence of coordinates $i,j$:

- $\text{Inf}^*_{i,j}(f) = \Pr_x[ f(x) \neq f(\mathbf{x}^{i \leftrightarrow j}) ]$.

Total influence:

- $\text{Inf}^*(f) = \sum_{i \neq j} \text{Inf}^*_{i,j}(f)$.

Influence of a set $S$ of coordinates:

- $\text{Inf}^*_S(f) = \Pr_{x, \pi \in S}[ f(x) \neq f(\pi x) ]$. 
Inf vs. Inf*
Properties of $\text{Inf}^*_{i,j}$ and $\text{Inf}^*$

- **Fact.** When $f$ is symmetric, $\text{Inf}^*(f) = 0$.
- **Fact.** $\text{Inf}^*_{i,j}(f) = \sum_{T : i, j \notin T} \left( \hat{f}(T \cup \{i\}) - \hat{f}(T \cup \{j\}) \right)^2$.
- **Theorem (KKL for $\text{Inf}^*$).** When $f$ is far from symmetric, there exist $i \neq j$ such that $\text{Inf}^*_{i,j}(f) = \Omega(\log(n)/n)$. [O’Donnell, Wimmer ’08]
Properties of \( \text{Inf}^*\mathcal{S} \)

- **Fact.** When \( f \) is \((n-k)\)-symmetric, there is a set \( J \) of size \(|J|=k\) s.t. \( \text{Inf}^{\lfloor n \rfloor \setminus \cup}(f) = 0 \).

- **Fact.** \( \text{Inf}^*\mathcal{S}(f) = \sum_{\mathcal{T}} \text{Var}_{\pi \in \mathcal{S}}( \hat{f}(\pi \mathcal{T}) ) \).

- **Lemma** (Monotonicity). \( \text{Inf}^*\mathcal{S}(f) \leq \text{Inf}^*\mathcal{S} \cup \mathcal{T}(f) \).

- **Lemma** (Subadditivity). If \(|S|,|T| \geq (1-\gamma)n\) then \( \text{Inf}^*\mathcal{S} \cup \mathcal{T}(f) \leq \text{Inf}^*\mathcal{S}(f) + \text{Inf}^*\mathcal{T}(f) + o(\gamma^{1/2}) \).
Properties of Inf*

**Theorem.** Let $f$ be $\varepsilon$-far from $(n-k)$-symmetric and let $P$ be a random $O(k^2)$-partition of $[n]$. Then whp every union $J$ of $k$ parts in $P$ satisfies $\text{Inf}^*[n]\cup(f) \geq \varepsilon/9$.

**Proof sketch.**

1. $F_{1/3} = \{S \subseteq [n]: \text{Inf}^*[n]\setminus S(f) < \varepsilon/3\}$ is $(k+1)$-intersecting.
2. If $F_{1/3}$ contains a set $S$ s.t. $|S| \leq 2k$, the bound holds.
3. Else, $F_{1/9} = \{S \subseteq [n]: \text{Inf}^*[n]\setminus S(f) < \varepsilon/9\}$ is $(2k+1)$-intersecting and the bound holds by the Intersection Theorem. ☐
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$$\inf^*[n \setminus (S \cap T)](f) = \inf^*[n \setminus S \cup [n \setminus T)](f) \leq \inf^*[n \setminus S](f) + \inf^*[n \setminus T](f) + \varepsilon/3 < \varepsilon$$
Properties of Inf*

**Theorem.** Let \( f \) be \( \epsilon \)-far from \((n-k)\)-symmetric and let \( P \) be a random \( O(k^2) \)-partition of \([n]\). Then whp every union \( J \) of \( k \) parts in \( P \) satisfies \( \text{Inf}^*[n\setminus J](f) \geq \epsilon/9 \).

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2. If \( F_{1/3} \) contains a set \( S \) s.t. \( |S| \leq 2k \), the bound holds.

W.h.p., \( S \) is shattered by \( P \) \( \Rightarrow \) \( J \cap S \leq k \) \( \Rightarrow \) \( J \notin F_{1/3} \).
Properties of $\text{Inf}^*$

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Each $J$ is $O(1/k)$-biased random set $\Rightarrow \Pr[J \in F_{1/9}] \leq k^{-2k}$. 

Discussion
Open Problems

- Which other results in the analysis of boolean functions can we extend to partial symmetry?
  - Friedgut’s junta theorem?
  - Structure of the Fourier spectrum?
- Can we use such extensions to prove the function isomorphism testing conjecture?
- In which other areas of TCS do partially symmetric functions appear?
  - Local reconstruction. [Alon, Weinstein ’12]
  - Active property testing. [Alon, Hod, Weinstein ’13]
Thanks!