Simple, Efficient and Neural Algorithms for Sparse Coding

Ankur Moitra (MIT)

joint work with Sanjeev Arora, Rong Ge and Tengyu Ma

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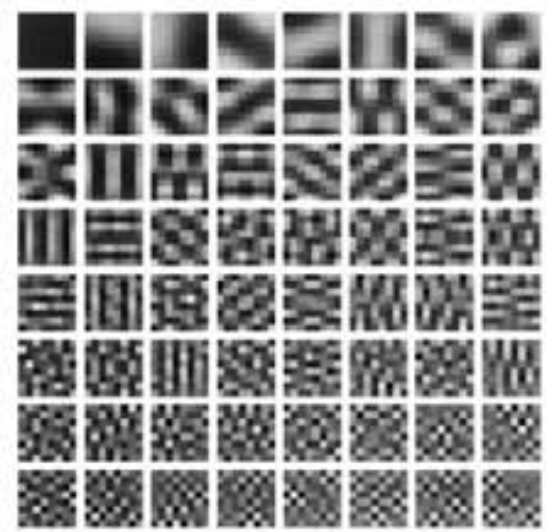

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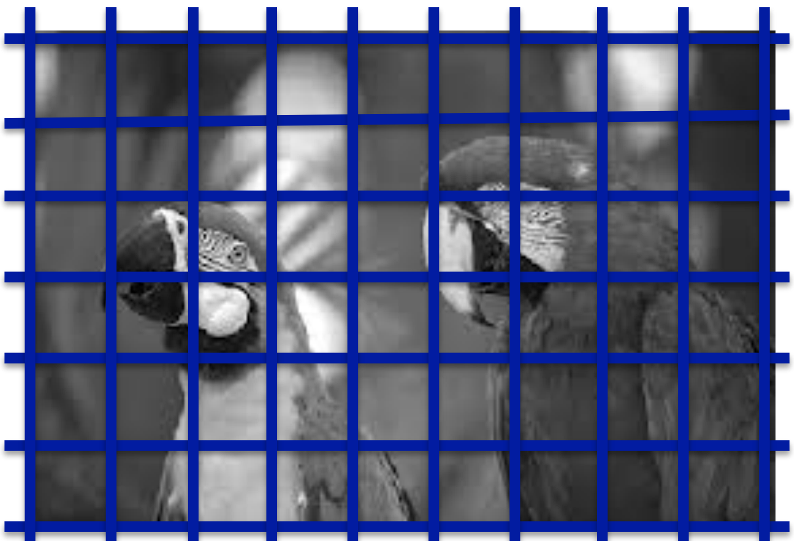

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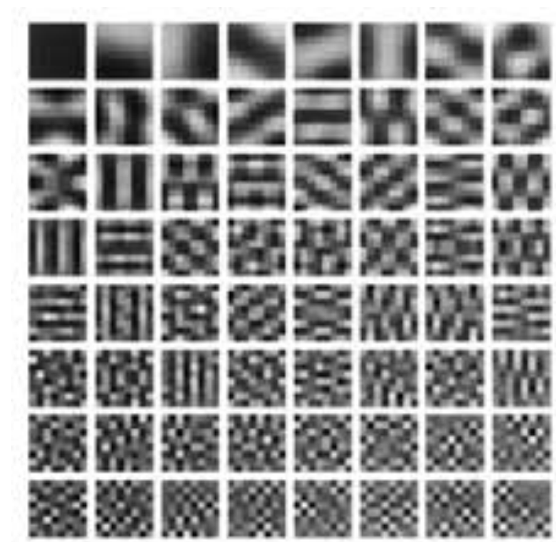

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Properties: localized, bandpass and oriented

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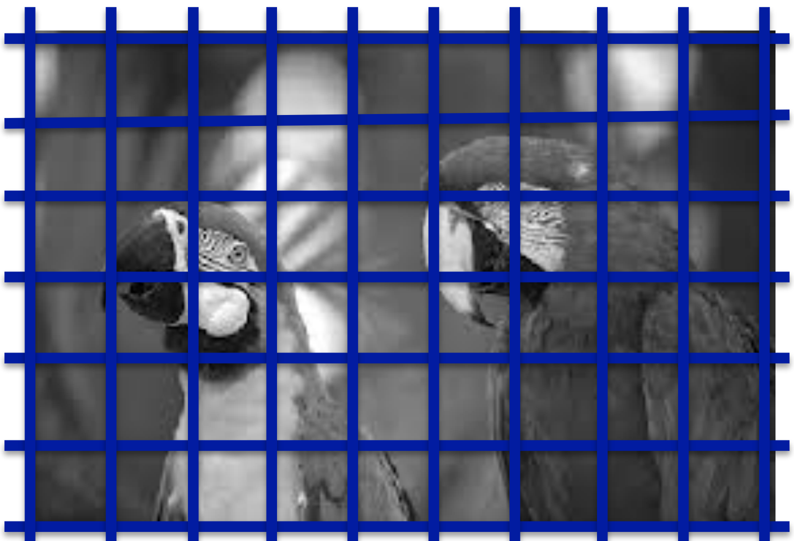

singular value decomposition

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Noisy!

Difficult to interpret!

(collection of vectors)

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NONCONVEX FORMULATIONS

Usual approach, minimize **reconstruction error**:

$$\min_{A,\, x^{(i)}\text{'s}} \sum_{i=1}^{p} \| b^{(i)} - A\, x^{(i)} \| + \sum_{i=1}^{p} L(x^{(i)})$$

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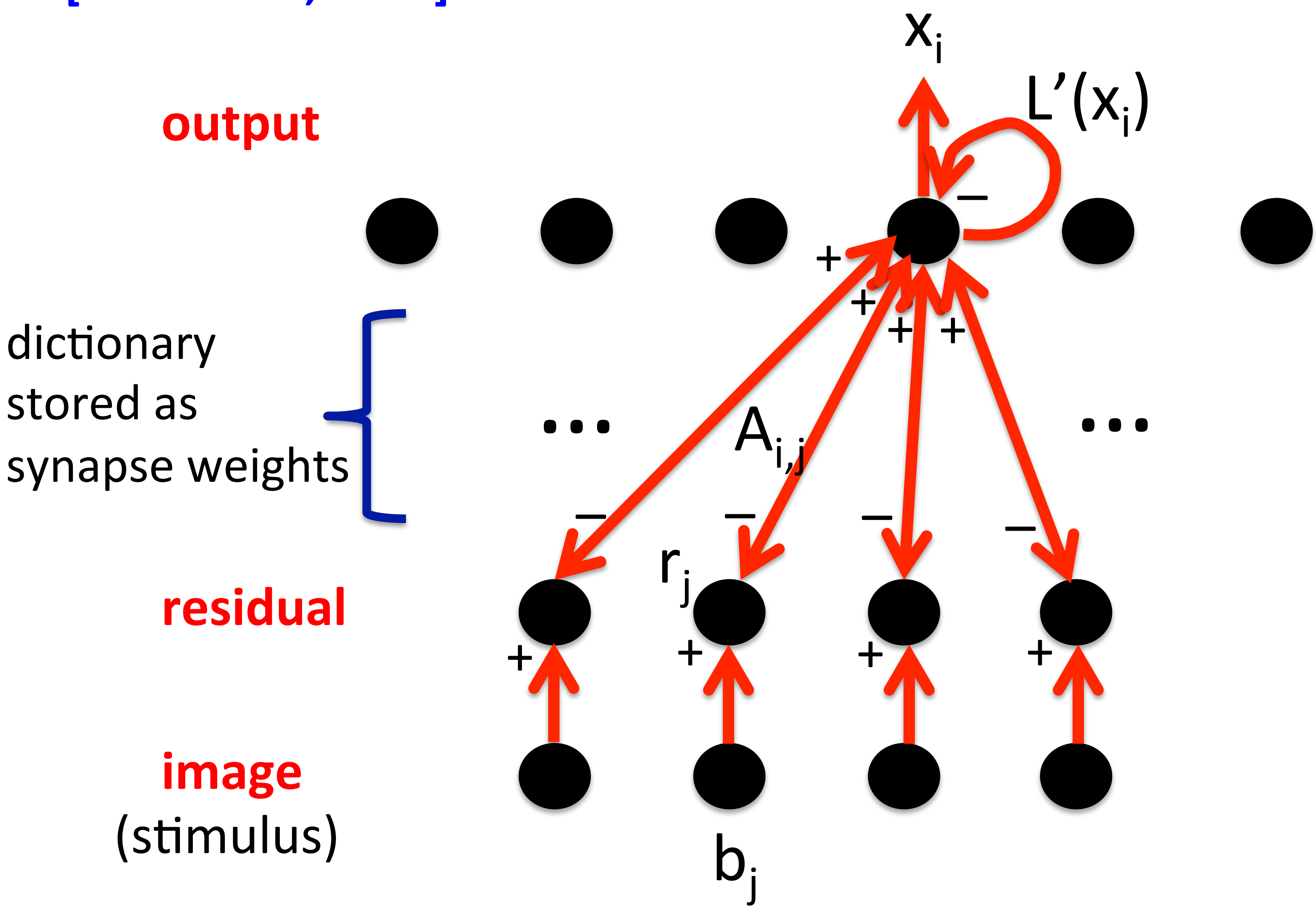
This optimization problem is **NP-hard**, can have many local optima; but **heuristics** work well empirically…

A NEURAL IMPLEMENTATION

[Olshausen, Field]:



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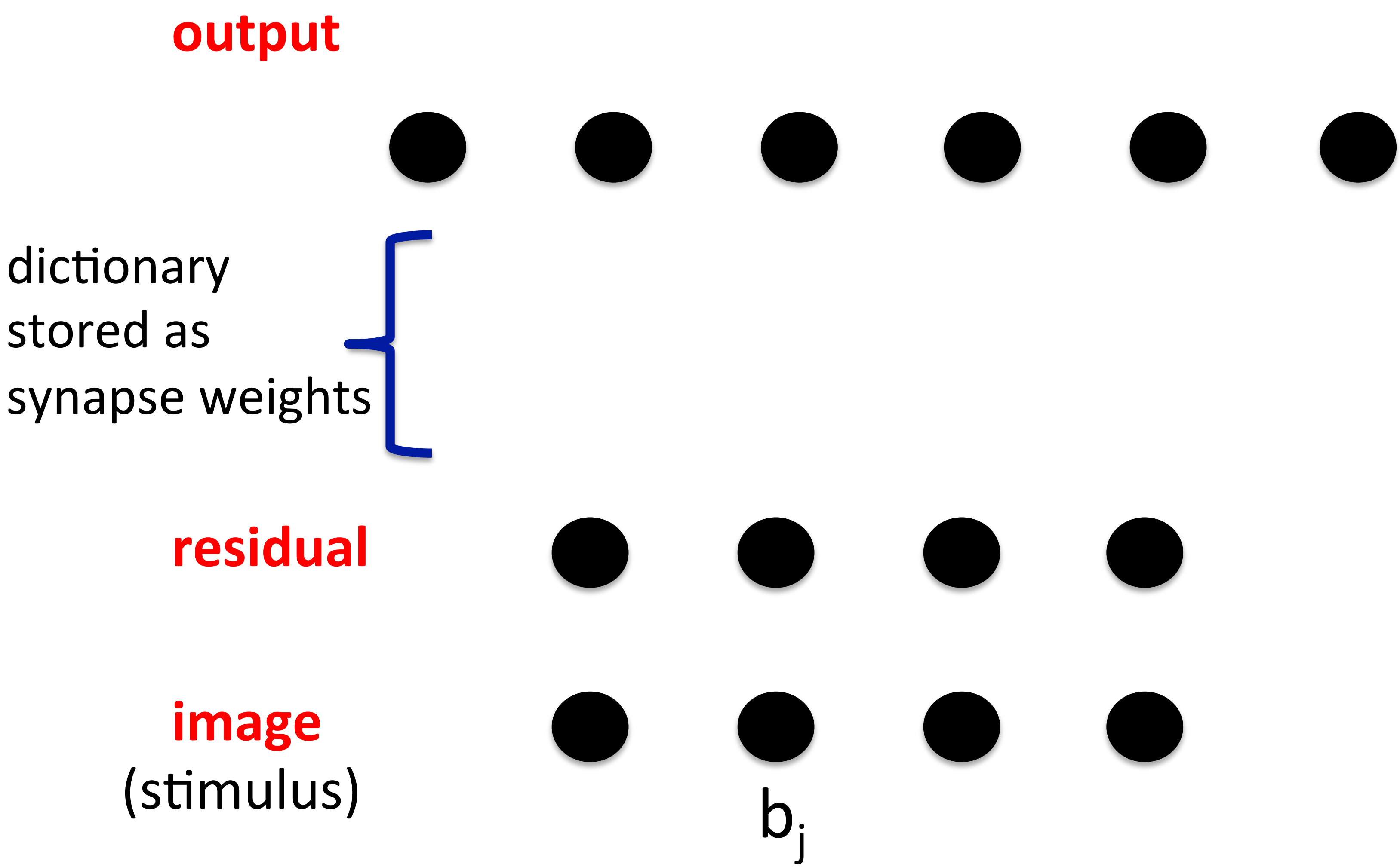

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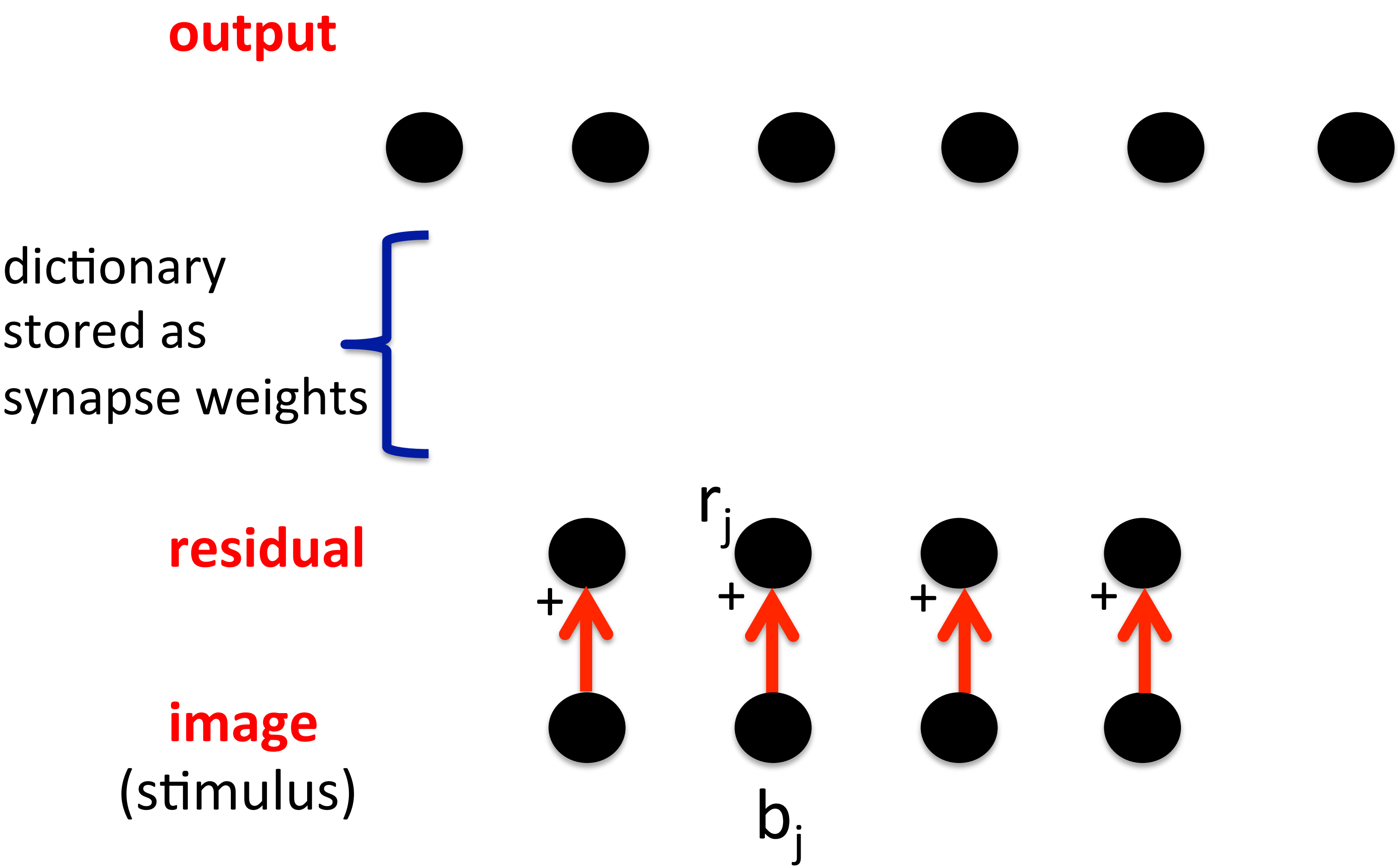

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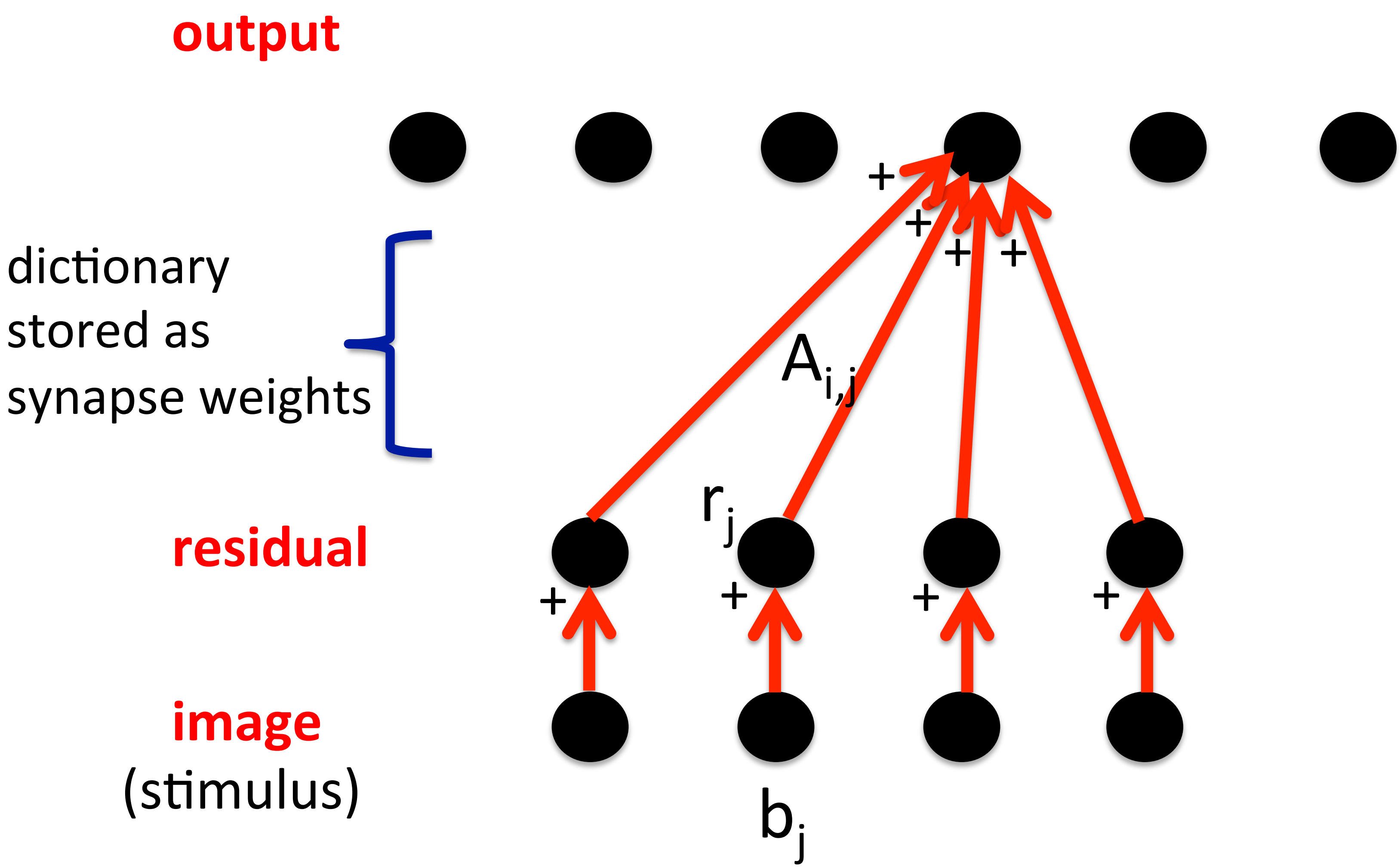

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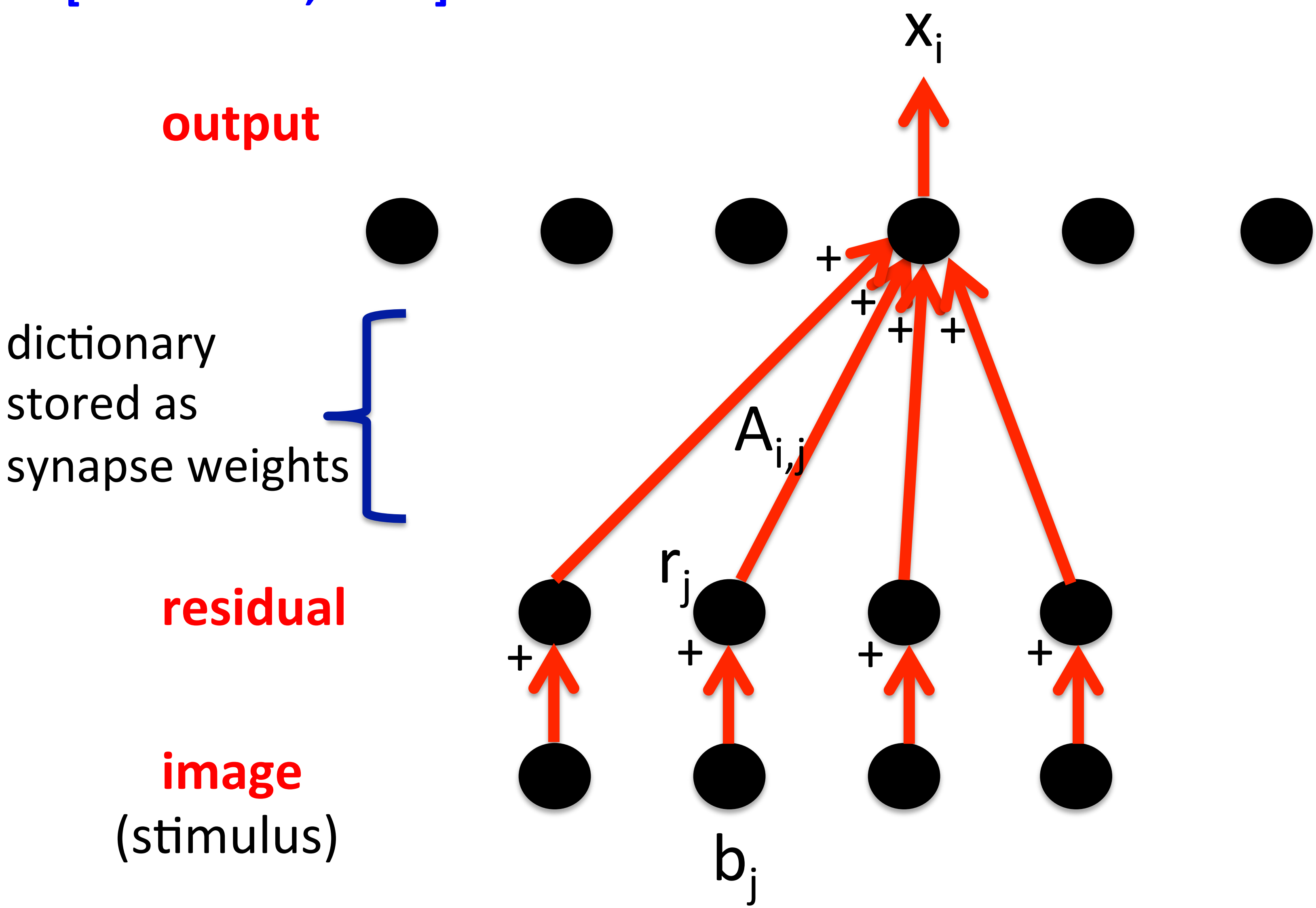

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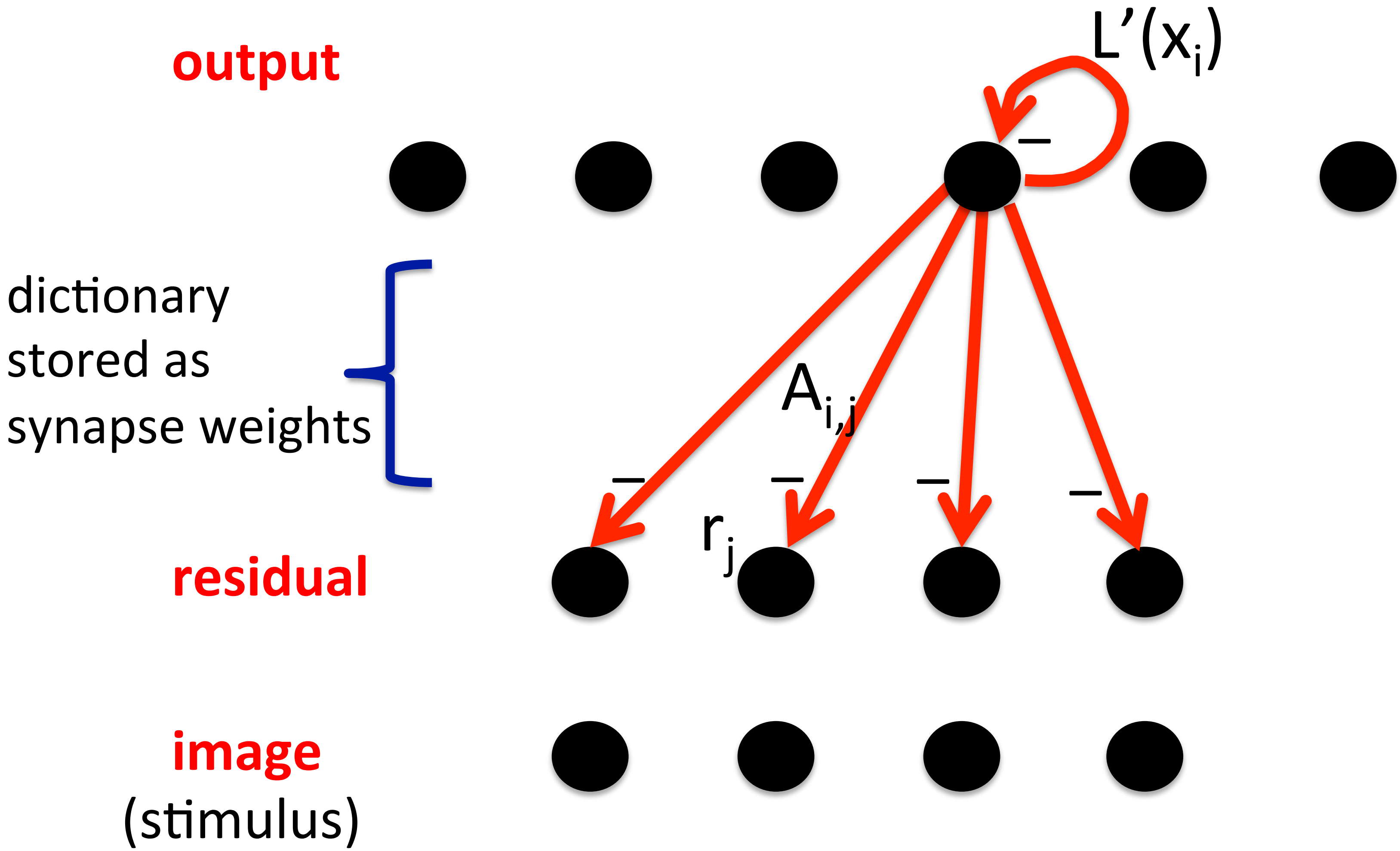

This network performs **gradient descent** by alternating between:

$$\textbf{(1)}\ r \longleftarrow b - Ax$$

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And A is updated a **Hebbian rule**

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Recent success in analyzing alternating minimization for matrix completion **[Jain, Netrapalli, Sanghavi]**, **[Hardt]**, phase retrieval **[Netrapalli, Jain, Sanghavi]**, robust PCA **[Anandkumar et al.]**, …

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[Barak, Kelner, Steurer '14]: works for overcomplete A up to sparsity roughly $n^{1-\varepsilon}$, but running time is **exponential** in accuracy

OUR RESULTS

Suppose $k \leq \sqrt{n}/\mu$ polylog(n) and $\|A\| \leq \sqrt{n}$ polylog(n)

Suppose $\widehat{A}$ that is column-wise δ-close to A for $\delta \leq 1/\text{polylog}(n)$

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We also give provable algorithms for **initialization** based on SVD

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Our results are based on a new **framework** for analyzing alternating minimization

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Alternate between the following steps (size q batches):

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Alternate between the following steps (size q batches):

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In contrast, previous (provable) algorithms might need to compute a new estimate **from scratch**, when new samples arrive

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In particular, the output is a thresholded, weighted sum of activations

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"neurons that fire together, wire together"

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The update to a weight $\hat{A}_{i,j}$ is the product of the activations at the residual layer and the decoding layer

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New Goal: Prove that (with high probability) the step **(2)** is **weakly** correlated with the gradient

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Their framework is about the **local geometry**, and ours is about the **direction of movement**

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This follows immediately from the usual proof…

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$$\hat{A}_j \longleftarrow \hat{A}_j + \xi\,(I - \hat{A}_j\hat{A}_j^T)A_j + \xi\,\mathbf{E}_R[\hat{A}_R\hat{A}_R^T]A_j + \text{error}$$

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Auxiliary Lemma: $\|\widehat{A} - A\| \leq 2$, remains true throughout if η is small enough and q is large enough

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E.g. we can remove the **systemic bias**, by carefully projecting out along the direction being updated

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Adjusting an iterative alg. can have subtle effects on its behavior

We can use our framework to **synthesize** new update rules

E.g. we can remove the **systemic bias**, by carefully projecting out along the direction being updated

(1) $\widehat{x}_j^{(i)} = \text{threshold}(\widehat{C}_j^T b^{(i)})$

where $\widehat{C}_j = [\text{Proj}_{\widehat{A}_j^\perp}(\widehat{A}_1), \text{Proj}_{\widehat{A}_j^\perp}(\widehat{A}_2), \ldots \widehat{A}_j \ldots \text{Proj}_{\widehat{A}_j^\perp}(\widehat{A}_m)]$

(2) $\widehat{A}_j \leftarrow \widehat{A}_j + \eta \sum_{i=1}^{q} (b^{(i)} - \widehat{C}_j \widehat{x}_j^{(i)}) \text{sgn}(\widehat{x}_j^{(i)})^T$

Any Questions?

Summary:

- **Online**, **local** and **Hebbian** algorithms for sparse coding that find a globally optimal solution (whp)

- Introduced a framework for analyzing iterative algorithms by thinking of them as trying to minimize an **unknown**, **convex** function

- The key is working with a generative model

- Is **computational intractability** really a barrier to a rigorous theory of neural computation?

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We give an initialization algorithm that outputs $\widehat{A}$ that is column-wise δ-close to A for $\delta \leq 1/\text{polylog}(n)$, $\|\widehat{A} - A\| \leq 2$

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output top eigenvector

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Key Lemma: If Ax = b and Ax′ = b′, then condition **(3)** is satisfied if and only if $\text{supp}(x) \cap \text{supp}(x') = \{j\}$

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Our initialization gets us to $\delta \leq 1/\text{polylog}(n)$, can be neurally implemented with **Oja's Rule**

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As a result, our bounds improve on existing algorithms in terms of **running time**, **sample complexity** and **sparsity** (all but SOS)